The Ergodic Conundrum

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Given a dynamical system in discrete time $f : M \to M$, where M is some metric space, it is our aim to give an imprecise interpretation to the following question:

"What is the probability of the physical system generated by f visiting a given set A?"

This question is two-fold: one may know the initial state of the system, x, or one may not.

When the system f admits an invariant probability μ defined on a σ-algebra of M the Birkhoff ergodic theorem implies that for μ-almost every point x and every measurable set A the frequencies of visits to the set A converge to, say, $\mathrm{freq}_A(x)$. Furthermore, if the dynamical system is ergodic then $\mathrm{freq}_A(x) = \mu(A)$, for μ-almost-every $x \in M$. This result might be seen as an argument in favour of the frequency interpretation of probability, however, as it is explained in [6], it encompasses a philosophical incoherence due to the hypothesis of the existence of an invariant probability.

Notwithstanding, and avoiding *a priori* defined invariant measures, for every point $x \in M$ we consider the usual limsup (resp., liminf) operator defined as $U_x(\varphi) = \limsup \frac{1}{n} \sum_{k=1}^{n} \varphi\left(f^k(x)\right)$ (resp., L_x) for every bounded function φ. This operator is monotone, positively homogeneous, translation-invariant and subadditive (resp. superadditive). Using a representation theorem in Föllmer and Schied [5] we can conclude that $U_x(\varphi) = \sup_{\nu \in F^*}\left(\int \varphi \, d\nu\right)$ and $L_x(\varphi) = \inf_{\nu \in F_*}\left(\int \varphi \, d\nu\right)$ where F^* and F_* are sets of finitely additive set functions. Furthermore, these operators generate two set functions, $u_x(A) := U_x(\mathbf{I}_A)$ and $l_x(A) := L_x(\mathbf{I}_A)$ which are submodular (resp. supermodular) using a characterisation result in Denneberg [3]. It is easy to see that these capacities are invariant under the dynamics of f (hence strengthening an existence result in [4]).

Therefore, if the initial state of the system is known, x say, we can define the "probability" of visiting a set A to be $\Pr_x(A) := [l_x(A), u_x(A)]$ and when it is not known $\Pr(A) := [\inf_{x \in M}\{l_x(A)\}, \sup_{x \in M}\{u_x(A)\}]$. In the case where there is a natural probability defined on a σ-algebra of the space M (e.g., a normalised Liouville measure, λ) then we can also define $\Pr(A) = [\int_M l_x(A)\, d\lambda, \int_M u_x(A)\, d\lambda]$. So far, we have not obtained any results based on these imprecise probabilities.

In a recent paper (see [1]) the authors proved a version of the ergodic theorem for lower and convex probabilities and these results can be applied to our set functions. Also, under different hypotheses, a version of the ergodic theorem for imprecise Markov chains is proved in [2].

Finally, we note that we can extend Poincaré recurrence theorem for superadditive set functions. However, we have not been able to prove, as yet, any type of Khintchine recurrence result nor any version of Kac's return time theorem.

References

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