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Imprecise Probabilities as a Semantics for Intuitive Probabilistic Reasoning

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Abstract

I prove a connection between the logical framework for intuitive probabilistic reasoning (IPR) introduced by Crane (2017) and sets of probabilities. More specifically, this connection provides a straightforward interpretation of imprecise probabilities as subjective credal states, giving a formal semantics for Crane's IPR proposal.

Keywords: imprecise probability, credence, credal state, intuitionistic logic, Martin-Löf type theory, evidence, subjective belief, justified belief

1. Introduction

In traditional subjectivist approaches to probability, degrees of belief are represented by a probability function (the Bayesian view (de Finetti, 1937)), as a non-additive belief function (as in the Dempster–Shafer theory (Shafer, 1976)), or more generally as a set of probabilities (in imprecise probability (Walley, 1990)). While the scope of imprecise probability extends beyond mere sets of probability, I focus here on the representation of subjective credal states by sets of probabilities, which is sufficient to demonstrate to core ideas of this theory while avoiding technical issues.[1] In what follows, I shall refer to this paradigm of subjective belief generically as the *IP framework*.

Here I show a connection between the above IP framework and a new framework for subjective probability judgment, which I call *intuitive probabilistic reasoning* (IPR). The IPR formalism was introduced by Crane (2017), has been previously applied in a formal logical system for the notion of 'typicality' (Crane and Wilhelm, 2019), and is expanded in Crane (forthcoming) as a logical framework for intuitive reasoning under uncertainty. The IPR formalism is independent of earlier frameworks of probability, introducing several new concepts, many of which lie beyond the scope of this brief note. Most germane to this article is its formal representation of subjective beliefs as mathematical objects other than conventional (imprecise) probability functions. I provide further technical preliminaries for this formalism in Section 2.

Within the IP paradigm above, subjective credences boil down to one or more functions that assign to any claim a numerical degree of belief. In IPR, by contrast, probabilities are formalized topologically, most precisely as a homotopy type (i.e., a topological space up to homotopy equivalence) whose structure captures the relationships among different pieces of evidence for a claim. I demonstrate this distinction with the following example.

Within the Bayesian paradigm, an agent's degree of belief about a claim A reflects a disposition toward betting on the truth of A. That is, an agent claiming $P(A) = p$ would pay up to $\$p$ to buy a contract that pays \$1 if A is true and \$0 if A is false, or would alternatively accept as little as $\$(1-p)$ to sell a contract which requires the agent to pay out \$1 if A is true and \$0 if A is false. Within the more general IP framework, in which the agent's credence is given by a set of probabilities $\mathscr{B}$, a betting quotient can be elicited by computing the lower and upper credences, respectively:

$$P_*(A) = \inf_{P\in\mathscr{B}} P(A) \quad \text{and}$$
$$P^*(A) = \sup_{P\in\mathscr{B}} P(A).$$

In the betting interpretation, the agent would be willing to pay up to $P(A)$ to buy a contract on A or accept payment as low as $P^*(A)$ to sell a contract on A, and would be unwilling to buy or sell for a price between $P(A)$ and $P^*(A)$.

In the IPR paradigm, an agent's belief about A is communicated as a judgment about evidence supporting the claim that 'A is probable'. IPR thus views probability as the basis for belief in the sense that belief in a claim is justified just in case it is deemed to be probable.[2] The 'probability of A' in this setting is interpreted qualitatively, in the same sense that the terms 'probable', 'likely', and such are invoked in everyday vernacular. More formally, the perspective of an individual agent is represented by $\mathbf{Bel}(A)$, a topological space (most precisely homotopy type) whose points correspond to pieces of evidence that would lead the agent to feel justified in making the judgment that 'A is probable', and whose topological structure (i.e., paths between points) represents the agent's judgment about the relationship between different pieces of evidence for A. The notation $\mathbf{Bel}(A)$ suggests the interpretation of a body of evidence for A, in the sense that the subject is justified in believing A whenever he

1. The results given below can be extended with little difficulty by interpreting every instance of 'P' as a Choquet capacity of order 2 instead of as a probability function.

2. N.B. This association between justified beliefs and probability judgments reflects an internal coherence of the agent rather than any claim about what is or is not justified or probable in an objective sense.

comes into possession of a piece of such evidence. While possession of any evidence for A may result in the same epistemic stance (namely belief in A), different justifications of belief $a : \mathbf{Bel}(A)$ and $a' : \mathbf{Bel}(A)$ constitute different beliefs, both formally and informally. Thus, instead of summarizing belief in A as $P(A) = p$ (a betting quotient), the agent reports a judgment of the form $a : \mathbf{Bel}(A)$, which is translated as 'A is probable on the basis of evidence a', or alternatively 'I believe A is probable because of a'.

More concretely, let A denote the claim that 'it is currently raining in New York City'. In the Bayesian paradigm, the agent who reports $P(A) = 0.20$ is willing to offer as high as 4-to-1 odds for a bet that loses if it is raining, or accept odds of 4-to-1 or greater for a bet that wins if it is raining. In the IPR paradigm, a concrete judgment of the form $a : \mathbf{Bel}(A)$ may be 'I believe it is raining in New York City because the weather report forecasted an 80% chance of rain today'. A different judgment of the same claim would be 'I believe it is raining because I saw a man carrying an umbrella'. In both cases, the claim ('it is raining') is the same but the content of the beliefs (a weather report versus visual confirmation) differ, and thus the beliefs themselves also differ.

Crane (2017) proposed the latter as a logical framework for intuitive, probabilistic reasoning because it aims to formalize the process by which individuals reason about uncertain claims by justifying, explaining, or otherwise rationalizing their beliefs, not by quoting a betting quotient. In the above statement, the agent offers a *reason* for believing A, i.e., that the weather report forecasted an 80% chance of rain. This reason is not presented as proof that it is raining but rather as justification for why the agent feels justified in believing that it is probably raining. The agent who saw someone carrying an umbrella also holds the belief that it is raining, but for a different reason. In IPR, these reasons are the content of probability judgments.

As I have presented them here, the IP and IPR frameworks concern probabilistic judgments of a different nature. The former summarizes belief quantitatively in terms of a willingness to bet, while the latter expresses belief qualitatively by providing the reason that an agent feels justified in believing something. The key difference lies in the content which is communicated by the belief: in IP, the precise odds are stated but the reason for stating those odds is not; in IPR, the precise reason is given but the odds are not.

Either representation may be appropriate depending on the circumstances, and I will not discuss here which of these representations is preferred in any given context. I only note that statements of the latter type, which report a qualitative reason for belief without a precise numerical quantification, are widespread in everyday common sense reasoning as well as in justifying scientific hypotheses, mathematical conjectures, legal arguments, and the like. Rather than focus on the differences between these formalisms, I instead show a connection between them in which the syntax of IPR, which regards $\mathbf{Bel}(A)$ as a body of evidence formalized as a topological space, is interpreted semantically as a set of probabilities, as in IP. These semantics suggest IPR as a potential logical foundation for imprecise probability that is autonomous from the traditional measure-theoretic foundation of probability.

1.1. Background and Prior Work

The relationship between probability and justified belief has a long history in epistemology, with earlier formal systems of belief given by Dempster–Shafer theory (Shafer, 1976), possibility theory (Zadeh, 1978; Dubois and Prade, 1988), belief revision theory (Alchourrón et al., 1985), and several others, e.g., Kyburg and Teng (2012). Each of these earlier treatments has its merits, and the reader familiar with this literature will easily see potential connections between the model proposed below and pieces of these earlier theories. In particular, the framework below can easily be recast in terms of Dempster–Shafer belief functions without any change in the final results. Further connections to these other frameworks, most notably the AGM postulates for belief revision, are left as a topic for future work.

Though intimately related, the treatment given below differs substantially from these earlier treatments, most obviously in its expression in terms of Martin-Löf type theory. To my knowledge, this is the first intuitionistic formalization of probability that is native to MLTT. There have been earlier category theoretic treatments of probability (Giry, 1982), but such work deals primarily with a reexpression of traditional probability within category theory. IPR, by contrast, begins with a primitive conception of probabilistic reasoning as a process that gives reasons for belief. This conception leads to formal axioms and, thus, the formal system of IPR, without any reference to the classical picture. Only after establishing the formalism of IPR does the connection to classical probability emerge, the topic of this paper.

Full understanding of the IPR formalism requires considerable technical background in intuitionistic logic (Brouwer, 1981; Heyting, 1971; Dummett, 2000), Martin-Löf type theory (Martin-Löf, 1984, 1987, 1996), and homotopy type theory (Univalent Foundations Program, 2013; Tsementzis, 2019; Kapulkin and Lumsdaine, 2011), for which there is insufficient space in this brief note. For further technical details about IPR, the reader is referred to the above references and to the original article (Crane, 2017). Before showing the connection between IP and IPR, I give a brief introduction to necessary formal aspects of IPR in the next section.

2. Preliminaries

I restrict here to the minimal fragment of Martin-Löf type theory (MLTT) necessary to communicate the main idea behind IPR. Readers unfamiliar with type-theoretic notation can safely interpret the syntax set-theoretically, reading '$A : \mathbf{Type}$' as '$A \in \mathbf{Set}$' (A is a set) and '$a : A$' as '$a \in A$' (a is an element of A). More important than the formalism in type theory is the distinction between the claim itself (denoted A) and an assertion of knowledge or belief regarding the claim (denoted $a : A$ or $a : \mathbf{Bel}(A)$, respectively).

In the fragment of MLTT used here, there are two primitive statements, which are called *judgments* and which correspond to *types* and *terms*, respectively:

judgment	formal meaning	interpretation
$A : \mathbf{Type}$	A is a Type	A is a claim
$a : A$	a is a term of type A	a is evidence for A

In the example of the previous section, A is the claim 'it is raining in New York City' and a term $a : A$ is a reason for believing A, e.g., 'because I see through the window that it is raining'. This interpretation of types as claims and terms as proofs has a long history at the intersection of logic and type theory and is called the Curry–Howard *propositions-as-types* interpretation (Curry and Feys, 1959; Howard, 1969).

In the IPR framework, we expand upon the Curry–Howard interpretation by associating to each claim A, whose terms are evidence that A is true, a claim $\mathbf{Bel}(A)$, whose terms are evidence that A is probable. So, in our running rain example, $a : A$ is a justification for believing that A is true whereas $a' : \mathbf{Bel}(A)$ is a justification for believing that A is probable or likely (e.g., because the weatherman forecasted an 80% chance of rain).

Finally, by the intuitionistic nature of MLTT, it is intended that judgments are interpreted relative to the context in which the agent makes it. Clearly, an agent who has knowledge of the 80% weather forecast will be in a different frame of mind with respect to the claim of rain than someone without knowledge of the forecast, and so the logic ought to reflect this difference in perspective. When expressing logical deductions formally, we write capital Greek letters $\Delta, \Gamma, \Xi, \ldots$ to denote a generic *context* in which a judgment is being made. Altogether, every judgment in MLTT is expressed in the form:

$$\text{Context } \Delta \quad \vdash \quad \text{Judgment } \mathscr{J},$$

which is interpreted to mean 'Judgment $\mathscr{J}$ is justified in Context Δ', where the judgment has the form of one of the two primitive judgments written above.

3. Axioms of Justified Belief

Crane (2017) suggests two core axioms of intuitive probabilistic reasoning about evidence:

(i) Knowledge justifies belief: for any claim A, $A \to \mathbf{Bel}(A)$.[3]

(ii) If A implies B then justified belief in A justifies belief in B: for any claims A and B,

$$(A \to B) \to (\mathbf{Bel}(A) \to \mathbf{Bel}(B)).$$

In the type-theoretic framework, the justification (i.e., the term) is a substantive component of any judgment, making the logical implications above insufficient for expressing these axioms. Believing that it is raining on the basis of a weather forecast is different than believing on the basis of seeing someone carrying an umbrella. To this end, the treatment of logical implication $A \to B$ marks one of the main differences between classical logic and intuitionistic type theory. Classically, the implication $A \to B$ is the material conditional $\neg A \vee B$, read as *if A then B*. In the constructive logic of type theory, however, the implication $A \to B$ requires that a witness for B can be explicitly constructed from any witness of A. Thus, in MLTT, logical implication $A \to B$ is formally a function $f : A \to B$ with domain A and codomain B which can be applied to any justification of A (i.e., $a : A$) in order to obtain a justification of B (i.e., $f(a) : B$).

The formal axiomatization of IPR (with **Type** written as **Claim** for emphasis) is expressed by the following three logical rules:

Formation rule:

$$\frac{\Delta \vdash A : \mathbf{Claim}}{\Delta \vdash \mathbf{Bel}(A) : \mathbf{Claim}} \quad (\mathbf{Bel}\text{-form}) \qquad (1)$$

Semi-formal: If 'A is true' is a valid claim in context Δ, then 'Belief in A' is a valid claim in Δ.

In the above semi-formal explanation, the description of A as being 'a valid claim' in context Δ reflects Martin-Löf's intuitionistic conception of meaning:

> "the meaning of a proposition [...] is determined by that which counts as a verification of it." (Martin-Löf, 1996, p. 27)

With this perspective, it is implicit to the assertion that 'A is a claim' that the claim has meaning, in the sense that the subject making the assertion knows what counts as a verification of the claim. It is in this sense that a claim is

3. Though not critical to the formalism it is worth nothing again that all claims in IPR are intuitionistic, and thus are relative to an agent's subjective disposition. In particular, I use the term 'knowledge' here in its non-technical sense, as when someone claims to know something, i.e., believes it to be true, on the basis of information. Knowledge is not restricted, as some epistemologists insist, to 'justified true belief', as in the formal system used here assumes neither a formal nor informal notion of truth independently of the agent's mental state.

considered valid. I will not belabor this point any further here; see Martin-Löf (1996) for further discussion.[4]

Introduction rule:

$$\frac{\Delta \vdash A : \mathbf{Claim}}{\Delta, a : A \vdash \mathtt{evid}_A(a) : \mathbf{Bel}(A)} \quad (\mathbf{Bel}\text{-intro}) \qquad (2)$$

Semi-formal: Proof that A is true provides justification for belief in A.

Elimination rule:

$$\frac{\Delta \vdash A : \mathbf{Claim} \quad \Delta \vdash C : \mathbf{Claim} \quad \Delta, a : A \vdash d(a) : C}{\Delta, x : \mathbf{Bel}(A) \vdash \mathtt{imp}_d(x) : \mathbf{Bel}(C)} \quad (\mathbf{Bel}\text{-elim}) \qquad (3)$$

Semi-formal: If a proof of C can be constructed from any proof of A in context Δ, then justification for believing C can be derived from any justification for believing A in Δ.

In some applications, it may also be appropriate to add the following axiom about belief in the logical contradiction, represented by the type $\mathbf{0} : \mathbf{Type}$ in MLTT.

0-rule:

$$\frac{\Delta\ \mathbf{ctx}}{\Delta, x : \mathbf{Bel}(\mathbf{0}) \vdash \sigma(x) : \mathbf{0}} \quad (\mathbf{Bel}\text{-}\mathbf{0}) \qquad (4)$$

Semi-formal: Belief in a vacuous claim is not justified in any context.

From these axioms alone it is possible to derive a number of intuitive results about the relationship between beliefs about individual claims and beliefs about their conjunction, disjunction, and negation. These results are deferred to Crane (2017). Here we focus on a specific interpretation of these axioms in terms of imprecise probabilities.

4. Connection to Imprecise Probabilities

For a set Ω, we let $\mathscr{P}(\Omega)$ denote the set of all probability spaces with base set Ω. In particular, the elements of $\mathscr{P}(\Omega)$ are triples $(\Omega, \mathscr{F}, P)$, where $\mathscr{F}$ is a σ-algebra on Ω and P is a (countably additive) probability function, i.e., a set function $\mathscr{F} \to [0,1]$ satisfying

(I) $P(\Omega) = 1$,

(II) $P(A) \geq 0$ for all $A \in \mathscr{F}$, and

(III) $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for mutually disjoint $A_1, A_2, \ldots \in \mathscr{F}$.

Condition (III) can be significantly weakened, e.g., to finite additivity or superadditivity, without changing the main result below.

We interpret Ω as a universe of possible states of the world, so that each claim A about the universe is represented by a subset $\tilde{A} \subseteq \Omega$ consisting of all worlds at which A holds. A particular $(\Omega, \mathscr{F}, P) \in \mathscr{P}(\Omega)$ can be interpreted as a subjective disposition toward the states of universe Ω, with $\tilde{A} \in \mathscr{F}$ indicating that the subject has a disposition about A (i.e., assigns a credence to $\tilde{A}$) and $P(\tilde{A})$ recording the credence. The function P represents credence from a particular frame of reference. A *credal state* is a set of credences $\mathscr{S} \subseteq \mathscr{P}(\Omega)$, and we say that an *agent occupies credal state* $\mathscr{S}$ just in case the agent's credence lies in $\mathscr{S}$. Singleton sets thus correspond to credal states that represent a precise credence P, but it is often possible to deduce how an agent would reason even without precise knowledge of their credence function. Knowing that their credences lie within a sufficiently broad credal state is often sufficient. Although this is an intuitive occurrence in everyday reasoning, formal treatments of subjective belief often require that an agent holds beliefs (precise or imprecise) about many more claims than necessary in a given situation. The connection to IPR may shed some light on this disconnect between formalism and practice.

In connecting credal states to IPR, a credal state $\mathscr{S}$ for which $\tilde{A} \in \mathscr{F}$ for every $(\Omega, \mathscr{F}, P) \in \mathscr{S}$ is one in which A is a valid claim, in the sense that an agent with any one of the dispositions in $\mathscr{S}$ assigns credence to A, and $\mathscr{S}$ for which $P(A) = 1$ for every $(\Omega, \mathscr{F}, P) \in \mathscr{S}$ is a credal state representing belief that 'A is true'. We extend this setup to include beliefs about the probability of A by adding a set $\widetilde{\mathbf{Bel}(A)}$ to $\mathscr{F}$ for each $\tilde{A} \in \mathscr{F}$ and specifying a threshold $1/2 < t \leq 1$ such that $P(\widetilde{\mathbf{Bel}(A)}) = 1$ whenever $P(\tilde{A}) \geq t$. A credal state $\mathscr{S}$ for which $P(A) \geq t$ for every $(\Omega, \mathscr{F}, P) \in \mathscr{S}$ thus represents the belief that 'A is probable', in the sense that an agent possessing any credence in $\mathscr{S}$ believes that A is sufficiently probable. An agent whose beliefs follow this thresholding protocol is said to obey the *Lockean thesis* (Foley, 2009).

We connect the semantics of IP with the syntax of IPR as follows. First, we write $\Delta\ \mathbf{ctx}$ to denote that Δ is a well-formed context according to the rules of MLTT. (For a full list of rules of MLTT, see either Univalent Foundations Program (2013) or the appendix of Kapulkin and Lumsdaine (2011).) For any $\tilde{A} \subseteq \Omega$, we define

$$\mathscr{F}_A := \{(\Omega, \mathscr{F}, P) \in \mathscr{P}(\Omega) \mid \tilde{A} \in \mathscr{F}\}$$

4. Notice that because $\mathbf{Bel}(A) : \mathbf{Claim}$ for every $A : \mathbf{Claim}$, it is permissible to iterate the **Bel** operator over beliefs, to obtain $\mathbf{Bel}(\mathbf{Bel}(A))$ (beliefs about beliefs), $\mathbf{Bel}(\mathbf{Bel}(\mathbf{Bel}(A)))$ (beliefs about beliefs about beliefs), and so on. The witnesses of these claims can be regarded as higher-order pieces of evidence, i.e., evidence of evidence, evidence of evidence of evidence, and so on, giving rise to a formal theory of higher-order belief. While this is an attractive feature of IPR with a number of potential applications, I don't discuss this any further here. Possible connections to Dorst's recent work on higher-order uncertainty (Dorst, forthcoming) are left for future research.

$$P_A := \{(\Omega, \mathscr{F}, P) \in \mathscr{P}(\Omega) \mid \tilde{A} \in \mathscr{F} \quad \text{and} \quad P(\tilde{A}) = 1\}$$

$$P_{\mathbf{Bel}(A)} := \{(\Omega, \mathscr{F}, P) \in \mathscr{P}(\Omega) \mid \tilde{A} \in \mathscr{F} \quad \text{and} \quad P(\tilde{A}) \geq t\},$$

for fixed $1/2 < t \leq 1$. When translating the syntax of IPR into the semantics of IP, we interpret the turnstile '$\vdash$' as '$\subseteq$' and commas on the left side of the turnstile as $\cap$. With this translation, the basic judgments of MLTT are interpreted as:

Syntax (IPR)	Semantics (IP)
Δ **ctx**	$\Delta \subseteq \mathscr{P}(\Omega)$
$\Delta \vdash A : \mathbf{Type}$	$\Delta \subseteq \mathscr{F}_A$
$\Delta \vdash a : A$	$\Delta \subseteq P_A$
$\Delta \vdash a' : \mathbf{Bel}(A)$	$\Delta \subseteq P_{\mathbf{Bel}(A)}$.

By this correspondence,

- a well-formed context Δ **ctx** in IPR corresponds to a credal state $\Delta \subseteq \mathscr{P}(\Omega)$. In the former, Δ is the perspective from which an agent makes a judgment; and in the latter, this perspective corresponds to the set of credences that represents the agent's possible credences;

- a judgment $\Delta \vdash A : \mathbf{Claim}$ in IPR is interpreted as 'A is a valid claim in context Δ', which in IP corresponds to a credence lying in a credal state $\Delta \subseteq \mathscr{F}_A$ for which every member assigns credence to A;

- the assertion $\Delta \vdash a : A$ in IPR is interpreted as 'a is evidence for A in context Δ', which in IP corresponds to a credence lying in a credal state $\Delta \subseteq P_A$ for which every member assigns credence 1 to A; and

- the assertion $a' : \mathbf{Bel}(A)$ in IPR is interpreted as 'a' is evidence that A is probable in context Δ', which in IP corresponds to a credence lying in a credal state $\Delta \subseteq P_{\mathbf{Bel}(A)}$ for which every member assigns sufficiently high credence to A, by the Lockean thesis.

To establish IP as a semantics for IPR, I prove soundness of the above translation in terms of the rules for MLTT and the type **Bel** defined above. The syntax of MLTT has additional type formers $\times$, $+$, and $\mathbf{0}$ corresponding to conjunction $\wedge$, disjunction $\vee$, and the contradiction $\perp$, respectively. In particular, for $A, B : \mathbf{Claim}$, $A \times B : \mathbf{Claim}$ is the claim 'A and B', $A + B : \mathbf{Claim}$ is 'A or B', and $\mathbf{0} : \mathbf{Claim}$ is the vacuous claim. The full translation from MLTT into set theory is given in the table below.

MLTT	set theory
Δ **ctx**	$\Delta \subseteq \mathscr{P}(\Omega)$
$A : \mathbf{Type}$	$\mathscr{F}_A$
$a : A$	P_A
$\vdash$	$\subseteq$
$A \times B$	$\tilde{A} \cap \tilde{B}$
$A + B$	$\tilde{A} \cup \tilde{B}$
$\mathbf{0}$	$\emptyset$

Theorem 1 *IP semantics is sound for IPR.*

To prove soundness, we interpret each of the rules for the $\times$, $+$, $\mathbf{0}$, and **Bel** types in IPR into the semantics of sets of probability functions and show that the rule holds. We begin by specifying the interpretation of the rules for contexts. In the following displays, the lefthand side shows the rules of MLTT (see Univalent Foundations Program, 2013) and the righthand side is the translation into IP according to the above protocol.

- **Structural rules**, $\bullet$-**ctx**:

Syntax	Semantics
$\dfrac{}{\bullet\ \mathbf{ctx}}$	$\dfrac{}{\mathscr{P}(\Omega) \subseteq \mathscr{P}(\Omega)}$

  Holds trivially: Every set is a subset of itself.[5]

- **Structural rules**, ext-**ctx**

Syntax	Semantics
$\dfrac{\Delta\ \mathbf{ctx} \qquad \Delta \vdash A : \mathbf{Claim}}{\Delta, x : A\ \mathbf{ctx}}$	$\dfrac{\Delta \subseteq \mathscr{P}(\Omega) \qquad \Delta \subseteq \mathscr{F}_A}{\Delta \cap P_A \subseteq \mathscr{P}(\Omega)}$

  By assumption $\Delta \subseteq \mathscr{P}(\Omega)$, and thus $\Delta \cap S \subseteq \Delta \subseteq \mathscr{P}(\Omega)$ for all other sets A. Instantiating $S = P_A$ gives the result.

- **Structural rules**, ax-**ctx**

Syntax	Semantics
$\dfrac{\Delta, a : A, \Xi\ \mathbf{ctx}}{\Delta, a : A, \Xi \vdash a : A}$	$\dfrac{\Delta \cap P_A \cap \Xi \subseteq \mathscr{P}(\Omega)}{\Delta \cap P_A \cap \Xi \subseteq P_A}$

  By assumption, $\Delta \cap P_A \cap \Xi$ is a set, and for any sets S and T it is always the case that $S \cap T \subseteq S$, yielding the result.

5. This rule states that there is an initial 'empty' context $\bullet$. In the semantics, the context places constraints on an agent's credal states, and thus this initial 'empty' context corresponds to a context without constraints, i.e., $\Delta \equiv \mathscr{P}(\Omega)$.

It follows from these structural rules for contexts that every context is a finite list of judgments of the form

$$(a_1 : A_1, \ldots, a_n : A_n)\ \mathbf{ctx},$$

which in our semantic interpretation translates to

$$P_{A_1} \cap \cdots \cap P_{A_n} \subseteq \mathscr{P}(\Omega).$$

Thus, in our semantic treatment, every context Δ can be expressed in the form

$$\Delta \equiv P_{A_1} \cap \cdots \cap P_{A_n} \tag{5}$$

for some finite list $A_1, \ldots, A_n \subseteq \Omega$. This specific representation will become useful when we prove soundness for the coproduct and **Bel**-types below.

We next prove soundness for the product type.

- **Product type**, formation rule:

Syntax	Semantics
$\Delta \vdash A : \mathbf{Claim}$	$\Delta \subseteq \mathscr{F}_A$
$\Delta \vdash B : \mathbf{Claim}$	$\Delta \subseteq \mathscr{F}_B$
$\Delta \vdash A \times B : \mathbf{Claim}$	$\Delta \subseteq \mathscr{F}_{A\times B}$

Let $(\Omega, \mathscr{F}, P) \in \Delta$ so that $\tilde{A}, \tilde{B} \in \mathscr{F}$. Then $\widetilde{A \times B} \equiv \tilde{A} \cap \tilde{B} \in \mathscr{F}$ because $\mathscr{F}$ is an algebra on Ω and thus is closed under intersection. It follows that $\Delta \subseteq \mathscr{F}_{A\times B}$, as claimed.

- **Product type**, introduction rule:

Syntax	Semantics
$\Delta \vdash A : \mathbf{Claim}$	$\Delta \subseteq \mathscr{F}_A$
$\Delta \vdash B : \mathbf{Claim}$	$\Delta \subseteq \mathscr{F}_B$
$\Delta, a : A, b : B \vdash (a, b) : A \times B$	$\Delta \cap P_A \cap P_B \subseteq P_{A\times B}$

By the formation rule, $\Delta \subseteq \mathscr{F}_A$ and $\Delta \subseteq \mathscr{F}_B$ implies $\Delta \subseteq \mathscr{F}_{A\times B}$, so that any $(\Omega, \mathscr{F}, P) \in \Delta$ has $\tilde{A} \cap \tilde{B} \in \mathscr{F}$ because $\mathscr{F}$ is an algebra. Furthermore, assume $(\Omega, \mathscr{F}, P) \in \Delta \cap P_A \cap P_B$ so that by finite additivity of probability functions, we have

$$\begin{aligned} P(\tilde{A} \cap \tilde{B}) &= P(\tilde{A}) + P(\tilde{B}) - P(\tilde{A} \cup \tilde{B}) \\ &= 1 + 1 - P(\tilde{A} \cup \tilde{B}). \end{aligned}$$

Finally, since $0 \leq P(S) \leq 1$ for all $S \in \mathscr{F}$, we must have $P(\tilde{A} \cup \tilde{B}) \leq 1$, implying $P(\tilde{A} \cap \tilde{B}) \geq 1$, and thus $P(\tilde{A} \cap \tilde{B}) = 1$, as claimed.

- **Product type**, elimination rule:

Syntax	Semantics
$\Delta \vdash A : \mathbf{Claim}$	$\Delta \subseteq \mathscr{F}_A$
$\Delta \vdash B : \mathbf{Claim}$	$\Delta \subseteq \mathscr{F}_B$
$\Delta, a : A, b : B \vdash C : \mathbf{Claim}$	$\Delta \cap \mathscr{F}_A \cap \mathscr{F}_B \subseteq \mathscr{F}_C$
$\Delta, a : A, b : B \vdash d(a, b) : C$	$\Delta \cap P_A \cap P_B \subseteq P_C$
$\Delta, z : A \times B \vdash \mathtt{split}_d(z) : C$	$\Delta \cap P_{A\times B} \subseteq P_C$

For $(\Omega, \mathscr{F}, P) \in \Delta \cap P_{A\times B} \subseteq P_{A\times B}$, we have

$$\min(P(\tilde{A}), P(\tilde{B})) \geq P(\tilde{A} \cap \tilde{B}) = 1;$$

whence $P(\tilde{A}) = P(\tilde{B}) = 1$ and $(\Omega, \mathscr{F}, P) \in \Delta \cap P_A \cap P_B$. By assumption, we have $\Delta \cap P_A \cap P_B \subseteq P_C$ so that $(\Omega, \mathscr{F}, P) \in P_C$, and thus $\Delta \cap P_{A\times B} \subseteq P_C$, as claimed.

Before we move on to discuss the coproduct type, we can use the rules for product type to deduce that $P_A \cap P_B = P_{A\times B}$ for all A, B : **Claim**. From this and the representation of contexts in the form (5), we can equivalently express any context as

$$\Delta \equiv P_{A_1 \times \cdots \times A_n}, \tag{6}$$

which can more compactly be written as

$$\Delta \equiv P_\Phi$$

for some $\tilde{\Phi} \in \mathscr{F}$, because $\tilde{A}_1 \cap \cdots \cap \tilde{A}_n \in \mathscr{F}$ whenever $\tilde{A}_1, \ldots, \tilde{A}_n \in \mathscr{F}$. This representation plays a role in our proof of soundness for the coproduct elimination rule.

- **Coproduct type**, formation rule:

Syntax	Semantics
$\Delta \vdash A : \mathbf{Claim}$	$\Delta \subseteq \mathscr{F}_A$
$\Delta \vdash B : \mathbf{Claim}$	$\Delta \subseteq \mathscr{F}_B$
$\Delta \vdash A + B : \mathbf{Claim}$	$\Delta \subseteq \mathscr{F}_{A+B}$

Let $(\Omega, \mathscr{F}, P) \in \mathscr{F}_A \cap \mathscr{F}_B$, then $\tilde{A}^c \in \mathscr{F}$, $\tilde{B}^c \in \mathscr{F}$, and $\tilde{A} \cap \tilde{B} \in \mathscr{F}$, because $\mathscr{F}$ is an algebra. Finally, by definition we have $\widetilde{A + B} \equiv \tilde{A} \cup \tilde{B} \equiv (\tilde{A}^c \cap \tilde{B}^c)^c \in \mathscr{F}$, because $\mathscr{F}$ is a σ-algebra and is closed under complementation and intersection. It follows that $(\Omega, \mathscr{F}, P) \in \mathscr{F}_{A+B}$.

- **Coproduct type**, left introduction rule:

Syntax	Semantics
$\Delta \vdash A : \mathbf{Claim}$	$\Delta \subseteq \mathscr{F}_A$
$\Delta \vdash B : \mathbf{Claim}$	$\Delta \subseteq \mathscr{F}_B$
$\Delta, a : A \vdash \mathtt{inl}(a) : A + B$	$\Delta \cap P_A \subseteq P_{A+B}$

By assumption, $\Delta \subseteq \mathscr{F}_A \cap \mathscr{F}_B$ implies that any $(\Omega, \mathscr{F}, P) \in \Delta$ assigns credence to A and B. The final premise $\Delta \cap P_A$ implies that $P(\tilde{A}) = 1$. By the preceding formation rule, we have $\tilde{A} \cup \tilde{B} \in \mathscr{F}$, and so P

assigns credence to it, and since P is increasing we must have $P(\tilde{A} \cup \tilde{B}) \geq P(\tilde{A}) = 1$; whence $P \in P_{A+B}$, as claimed.

The same argument carries through to prove the right introduction rule for the coproduct type.

- **Coproduct type**, elimination rule:

Syntax	Semantics
$\Delta \vdash A : \textbf{Claim}$	$\Delta \subseteq \mathscr{F}_A$
$\Delta \vdash B : \textbf{Claim}$	$\Delta \subseteq \mathscr{F}_B$
$\Delta, a : A, b : B \vdash C : \textbf{Claim}$	$\Delta \cap P_A \cap P_B \subseteq \mathscr{F}_C$
$\Delta, a : A \vdash d_l(a) : C$	$\Delta \cap P_A \subseteq P_C$
$\Delta, b : B \vdash d_r(b) : C$	$\Delta \cap P_B \subseteq P_C$
$\Delta, z : A + B \vdash \mathtt{case}_{d_l,d_r}(z) : C$	$\Delta \cap P_{A+B} \subseteq P_C$

Here we use the representation in (6) to express $\Delta \equiv P_\Phi$ for some $\Phi \equiv A_1 \times \cdots \times A_n$, so that the third and fourth assumptions and the conclusion on the righthand side, respectively, become

$$P_{\Phi \times A} \subseteq P_C, \quad P_{\Phi \times B} \subseteq P_C \quad \text{and} \quad P_{\Phi \times (A+B)} \subseteq P_C.$$

By assumption, we have $P_{\Phi \times A} \subseteq P_C$. Thus, any P that satisfies $P(\tilde{\Phi} \cap \tilde{A}) = 1$ must also satisfy $P(\tilde{C}) = 1$, which is possible only if $\tilde{\Phi} \cap \tilde{A} \subseteq \tilde{C}$. For suppose that there is some $\omega \in \tilde{\Phi} \cap \tilde{A}$ for which $\omega \notin \tilde{C}$. Then there is a measurable space $(\Omega, \mathscr{F}_\omega, P_\omega)$ with σ-algebra $\mathscr{F}_\omega = \{\Omega, \emptyset, \{\omega\}, \Omega \setminus \{\omega\}\}$ and P_ω the atomic measure at $\{\omega\}$ (i.e., $P_\omega(\{\omega\}) = 1$). With $\omega \in \tilde{\Phi} \cap \tilde{A}$, it follows that $P_\omega(\tilde{\Phi} \cap \tilde{A}) \geq P_\omega(\{\omega\}) = 1$ and $P_\omega(\tilde{C}) = 0$, contradicting the assumption. By applying an analogous argument to the fourth assumption, we must have $\tilde{\Phi} \cap \tilde{A} \subseteq \tilde{C}$.

Finally, note that $\tilde{\Phi} \cap (\tilde{A} \cup \tilde{B}) \equiv (\tilde{\Phi} \cap \tilde{A}) \cup (\tilde{\Phi} \cap \tilde{B})$, so that the conclusion reads

$$P_{(\Phi \times A)+(\Phi \times B)} \subseteq P_C.$$

Now, suppose $(\Omega, \mathscr{F}, P) \in P_{(\Phi \times A)+(\Phi \times B)}$ so that $P((\tilde{\Phi} \cap \tilde{A}) \cup (\tilde{\Phi} \cap \tilde{B})) = 1$. Then by the preceding argument we have $\tilde{\Phi} \cap \tilde{A} \subseteq \tilde{C}$ and $\tilde{\Phi} \cap \tilde{B} \subseteq \tilde{C}$, which implies

$$(\tilde{\Phi} \cap \tilde{A}) \cup (\tilde{\Phi} \cap \tilde{B}) \subseteq \tilde{C}.$$

It follows that

$$1 = P((\tilde{\Phi} \cap \tilde{A}) \cup (\tilde{\Phi} \cap \tilde{B})) \leq P(\tilde{C});$$

whence, $P(\tilde{C}) = 1$ and $(\Omega, \mathscr{F}, P) \in P_C$, as claimed.

We next discuss the **0** type.

- **0-type**, formation rule:

Syntax	Semantics
Δ **ctx**	$\Delta \subseteq \mathscr{P}(\Omega)$
$\Delta \vdash \mathbf{0} : \textbf{Claim}$	$\Delta \subseteq \mathscr{F}_{\mathbf{0}}$

As any σ-algebra contains $\tilde{\mathbf{0}} = \emptyset$ it is immediate that $\mathscr{P}(\Omega) = \mathscr{F}_{\mathbf{0}} = \{(\Omega, \mathscr{F}, P) \in \mathscr{P}(\Omega) \mid \emptyset \in \mathscr{F}\}$ and the conclusion follows.

- **0 type**, elimination rule:[6]

Syntax	Semantics
$\Delta \vdash A : \textbf{Claim}$	$\Delta \subseteq \mathscr{F}_A$
$\Delta, x : \mathbf{0} \vdash \mathtt{efq}_A(x) : A$	$\Delta \cap P_{\mathbf{0}} \subseteq P_A$

The subset $P_{\mathbf{0}} \subseteq \mathscr{P}(\Omega)$ consists of all $(\Omega, \mathscr{F}, P)$ that assign credence 1 to $\emptyset$. Since $P(\emptyset) = 0$ for any probability function, it follows that $P_{\mathbf{0}} = \emptyset$ and $\Delta \cap P_{\mathbf{0}} \equiv \emptyset \subseteq P_A$ holds trivially.

Finally, for the **Bel**-type.

- **Belief type**, formation rule:

Syntax	Semantics
$\Delta \vdash A : \textbf{Claim}$	$\Delta \subseteq \mathscr{F}_A$
$\Delta \vdash \textbf{Bel}(A) : \textbf{Claim}$	$\Delta \subseteq \mathscr{F}_{\textbf{Bel}(A)}$

We require that $\widetilde{\textbf{Bel}(A)}$ is a measurable set whenever $\tilde{A}$ is, so that the conclusion immediately follows by our extended definition of $\mathscr{P}(\Omega)$.

- **Belief type**, introduction rule:

Syntax	Semantics
$\Delta \vdash A : \textbf{Claim}$	$\Delta \subseteq \mathscr{F}_A$
$\Delta, a : A \vdash \mathtt{evid}_A(a) : \textbf{Bel}(A)$	$\Delta \cap P_A \subseteq P_{\textbf{Bel}(A)}$

As any $(\Omega, \mathscr{F}, P) \in \Delta \cap P_A$ must satisfy $P(A) = 1$ and it is required that $\widetilde{\textbf{Bel}(A)} \supseteq \tilde{A}$, we must have $P(\widetilde{\textbf{Bel}(A)}) \geq P(\tilde{A}) = 1$, and $(\Omega, \mathscr{F}, P) \in P_{\textbf{Bel}(A)}$, as claimed.

- **Belief type**, elimination rule:

Syntax	Semantics
$\Delta \vdash A : \textbf{Claim}$	$\Delta \subseteq \mathscr{F}_A$
$\Delta \vdash B : \textbf{Claim}$	$\Delta \subseteq \mathscr{F}_B$
$\Delta, a : A \vdash d(a) : B$	$\Delta \cap P_A \subseteq P_B$
$\Delta, x : \textbf{Bel}(A) \vdash \mathtt{imp}_d(x) : \textbf{Bel}(B)$	$\Delta \cap P_{\textbf{Bel}(A)} \subseteq P_{\textbf{Bel}(B)}$

6. Here `efq` stands for *ex falso quodlibet* ("from falsehood, anything follows"). Formally, this rule says that given any A : **Claim** and a proof $x : \mathbf{0}$ of the contradiction it is possible to construct a proof $\mathtt{efq}_A(x) : A$.

By (6), we can express the second assumption as $P_{\Phi\cap A} \subseteq P_B$ for some $\tilde{\Phi} \subseteq \Omega$, from which it follows that $\tilde{\Phi} \cap \tilde{A} \subseteq \tilde{B}$ by an argument already given above when proving the elimination rule for the coproduct type. Thus, we can rewrite the conclusion as

$$P_{\Phi\times\mathbf{Bel}(A)} \subseteq P_{\mathbf{Bel}(B)}.$$

Let $(\Omega, \mathscr{F}, P) \in P_{\Phi\times\mathbf{Bel}(A)}$. Then $P(\tilde{\Phi} \cap \widetilde{\mathbf{Bel}(A)}) = 1$, and in particular $P(\tilde{\Phi}) = 1$ and $P(\widetilde{\mathbf{Bel}(A)}) = 1$, implying that $P(\tilde{A}) \geq 1 - t$.

By definition we have

$$\begin{aligned} P(\tilde{\Phi} \cap \tilde{A}) &= P(\tilde{\Phi}) + P(\tilde{A}) - P(\tilde{\Phi} \cup \tilde{A}) \\ &\geq P(\tilde{\Phi}) + (1-t) - 1 \\ &= 1 - t. \end{aligned}$$

By assumption, we have $P(\tilde{B}) \geq P(\tilde{\Phi} \cap \tilde{A}) \geq 1 - t$, and $P(\widetilde{\mathbf{Bel}(B)}) = 1$ by definition.

- **Belief type**, **0** rule:

Syntax	Semantics
$\dfrac{\Delta\ \mathbf{ctx}}{\Delta, x : \mathbf{Bel}(\mathbf{0}) \vdash \sigma(x) : \mathbf{0}}$	$\dfrac{\Delta \subseteq \mathscr{P}(\Omega)}{\Delta \cap P_{\mathbf{Bel}(\mathbf{0})} \subseteq P_{\mathbf{0}}}$

Every probability function assigns 0 credence to $\tilde{\mathbf{0}} \equiv \emptyset$. Thus there does not exist any P for which $P(\emptyset) \geq 1 - t$, and it follows that $\Delta \cap P_{\mathbf{Bel}(\mathbf{0})} \subseteq P_{\mathbf{0}} \equiv \emptyset$, as required.

5. Additional Results and Further Concepts

The above framework facilitates a number of intuitive probabilistic notions, such as independence and conditional probability, and elicits several additional results as theorems. Though proofs of these theorems are straightforward, they require a working knowledge of MLTT and homotopy type theory (HoTT) which I have not assumed in this semi-expository piece. I instead state these consequences below and discuss their significance in the broader scope of IPR and IP. A more complete treatment of these results can be found in Crane (2017, 2019+).

The first few theorems concern the dependence between beliefs in conjunction and disjunction, namely

$$\begin{aligned} &\mathbf{Bel}(A \times B) \to \mathbf{Bel}(A) \times \mathbf{Bel}(B) \to \\ &\quad \to \mathbf{Bel}(A) + \mathbf{Bel}(B) \to \mathbf{Bel}(A + B). \end{aligned} \tag{7}$$

In the language of justified belief this result can be understood as follows. Belief in a conjunction warrants belief in each of the conjuncts ($\mathbf{Bel}(A \times B) \to \mathbf{Bel}(A) \times \mathbf{Bel}(B)$) and (separate) beliefs in any two claims warrants belief in either one of those claims ($\mathbf{Bel}(A) \times \mathbf{Bel}(B) \to \mathbf{Bel}(A) + \mathbf{Bel}(B)$). Finally, belief in either of two claims warrants belief in at least one of the claims ($\mathbf{Bel}(A) + \mathbf{Bel}(B) \to \mathbf{Bel}(A + B)$).

Note that the arrows do not reverse in general. To understand why, pay close attention to the interpretation of judgments in terms of *justified belief*, and in particular the role of justification as the content of belief. A claim that one believes A and B jointly, i.e., a judgment $a : \mathbf{Bel}(A \times B)$, is a claim that a provides coherent evidence for A and B, in the sense that a constitutes justification for believing both claims. A claim that one believes A and B separately, i.e., $a : \mathbf{Bel}(A) \times \mathbf{Bel}(B)$, is weaker, as it asserts that one has justification for believing A and justification for believing B, but that the individual justifications may not cohere into a single piece of evidence that justifies belief in both.

The lottery paradox is a well known instance where the reverse implication is known to fail. To wit, consider a lottery with 1000 equiprobable tickets and A_i : **Claim** denoting the claim that the i ticket is the winner. A Lockean with threshold $t = 0.99$ is justified in believing that each of the tickets is a loser as each has a 0.999 marginal probability of losing, but this does not warrant belief that all tickets will lose, i.e., $\mathbf{Bel}(\neg A_1 \times \cdots \times \neg A_{1000})$. This result is made formal in the following consequence for universal and existential quantification in IPR:

$$\begin{aligned} &\mathbf{Bel}(\prod_{a:A} B(a)) \to \prod_{a:A} \mathbf{Bel}(B(a)) \\ &\sum_{a:A} \mathbf{Bel}(B(a)) \to \mathbf{Bel}(\sum_{a:A} B(a)). \end{aligned}$$

As above, the arrows to not reverse in general.

5.1. Conditional Belief

Conditional beliefs are formally defined in IPR as a dependent type. For any A, B : **Claim**, there is a type former $\mathbf{Bel}(B \mid -) : \mathbf{Bel}(A) \to$ **Claim** that maps each reason for believing A to a conditional claim for believing B. More precisely, this type is defined for each $a : \mathbf{Bel}(A)$ by

$$\mathbf{Bel}(B \mid a) :\equiv \sum_{x:\mathbf{Bel}(A\times B)} (\mathtt{imp}_{\mathtt{pr}_A}(x) =_{\mathbf{Bel}(A)} a), \tag{8}$$

where $\mathtt{imp}_{\mathtt{pr}_A} : \mathbf{Bel}(A \times B) \to \mathbf{Bel}(A)$ is the canonical map derived from the first implication in (7) and, in general, $a =_A a'$: **Claim** denotes the claim that $a : A$ and $a' : A$ are identical pieces of evidence for A : **Claim**.

We have not covered the necessary concept of identity types to explain this definition formally, but the concept has the following intuitive interpretation. Each term b : $\mathbf{Bel}(B \mid a)$ is a composite belief consisting of a justification $x : \mathbf{Bel}(A \times B)$ for joint belief in both A and B and a claim that this justification is coherent with the conditioning piece of evidence $a : \mathbf{Bel}(A)$, that is, the support of A implied by x, via $\mathtt{imp}_{\mathtt{pr}_A}(x) : \mathbf{Bel}(A)$, is identical to the support for

A provided by $a : \mathbf{Bel}(A)$. This is precisely what it means to have a conditional belief in B given a belief $a : \mathbf{Bel}(A)$ in A. By appeal to HoTT's univalence axiom (Univalent Foundations Program, 2013), one can prove an analog to the classical law of total probability,

$$\mathbf{Bel}(A \times B) \simeq \sum_{a:\mathbf{Bel}(A)} \mathbf{Bel}(B \mid a),$$

where '$\simeq$' stands for homotopy equivalence between the homotopy types representing the claims on left and right.

5.2. Independence

By analogy to the conventional definition of independence for probability functions, i.e., $P(A \cap B) = P(A)P(B)$, we define independence of claims A, B : **Claim** in IPR by

$$\mathbf{Bel}(A \times B) \simeq \mathbf{Bel}(A) \times \mathbf{Bel}(B). \tag{9}$$

By comparison to (7), independence of A and B is a strong assertion which implies that justified belief in each of A and B individually warrants joint belief in A and B. Indeed, that suits the intuitive notion of independence: as any reasons for believing claims that are independent cannot interfere with one another, the two separate reasons are sufficient for believing the joint claim. Further extensions to conditional independence and connections to other notions from traditional probability calculus are possible and left for future work.

6. Concluding Remarks

I have outlined a new framework for intuitive probabilistic reasoning (Crane, 2017, forthcoming). I believe that Sections 2, 3, and 5 establish this system as a standalone intuitive account for the kind of qualitative probabilistic reasoning that individuals engage in instinctively. The proof of soundness in Section 4 establishes a formal connection between this new formalism and the traditional representation of degrees of belief via probability functions. Beyond these possible connections to traditional probability and variants of imprecise probability, IPR seems a system worthy of study in its own right, and it is the route taken in follow up work by Crane and Wilhelm (2019) and Crane (forthcoming).

In light of the connection to traditional probabilities, however, it is worthwhile to explore how the IPR framework might be useful for studying general properties about models in IP. In particular, IPR could be explored as a formal foundation of imprecise probability, in the sense that many formal statements in IP can be interpreted into IPR. The existence of such a foundation offers at least two potential benefits:

1. It provides a standard of rigor which allows general theorems in IP to be established by abstracting to IPR and proving a number of special cases all at once. The amenability of MLTT, and thus IPR, to computerized proof assistants, such as Coq, offers another potential practical benefit when attempting such proofs.

2. IPR as a foundation for IP would establish imprecise probability as autonomous from classical probability theory, making precise the concept of what is an imprecise probability, without the need to refer to traditional 'precise' probabilities. In particular, sets of probabilities are just one possible instantiation of what might rightly be called 'imprecise probability', and the formalism of IPR provides a general abstract setting in which to explore the boundaries of this notion.

References

C. E. Alchourrón, P. Gärdenfors, and D. Makinson. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic*, 50:510–530, 1985.

L. E. J. Brouwer. *Brouwer's Cambridge Lectures on Intuitionism*. Cambridge University Press, 1981.

H. Crane. The logic of probability and conjecture. *Researchers.One*, 2017. URL https://www.researchers.one/article/2018-08-5.

H. Crane. *Intuitive Probabilistic Reasoning*. forthcoming. Book in progress.

H. Crane and I. Wilhelm. The logic of typicality. In V. Allori, editor, *Statistical Mechanics and Scientific Explanation: Determinism, Indeterminism and Laws of Nature*. 2019.

H. B. Curry and R. Feys. *Combinatory Logic*, volume I of *Studies in Logic and the Foundations of Mathematics*. 1959.

B. de Finetti. La prévision: ses lois logiques, ses sources, subjectives. *Annales de l'Institut Henri Poincaré*, 7:1–68, 1937.

K. Dorst. Higher-order uncertainty. In M. Skipper Rasmussen and A. Steglich-Peterson, editors, *Higher-Order Evidence: New Essays*. Oxford University Press, forthcoming.

D. Dubois and Henri Prade. *Possibility Theory, An Approach to Computerized Processing of Uncertainty*. Plenum, New York, 1988.

M. Dummett. *Elements of Intuitionism*, volume 39 of *Oxford Logic Guides*. Clarendon Press, 2nd edition, 2000.

R. Foley. Belief, degrees of belief, and the Lockean thesis. In F. Huber and C. Schmidt-Petri, editors, *Degrees of Belief*, pages 37–47. Springer, Dordrecht, 2009.

M. Giry. A categorical approach to probability theory. In B. Banaschewski, editor, *Categorical Aspects of Topology and Analysis*, pages 68–85. Springer, 1982. doi:10.1007/BFb0092872.

A. Heyting. *Intuitionism: An introduction*. Studies in logic and the foundations of mathematics. North-Holland Pub. Co, 3rd edition, 1971.

W. A. Howard. The formulae-as-types notion of construction. In Jonathan P. Seldin and Roger Hindley J., editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 479–490. 1969.

C. Kapulkin and P. L. Lumsdaine. The simplicial model of univalent foundations (after Voevodsky), 2011. URL https://arxiv.org/abs/1211.2851.

H. E. Kyburg, Jr. and C. Teng. The logic of risky knowledge, reprised. *Int. J. Approx. Reason.*, 53:274–285, 2012.

P. Martin-Löf. Intuitionistic type theory. volume 1 of *Studies in Proof Theory*. Bibliopolis, 1984. URL http://www.cse.chalmers.se/~peterd/papers/MartinL%C3%B6f1984.pdf. Lecture Notes.

P. Martin-Löf. Truth of a proposition, evidence of a judgment, validity of a proof. *Synthese*, 73:407–420, 1987.

P. Martin-Löf. On the meanings of the logical constants and the justifications of the logical laws. *Nordic Journal of Philosophical Logic*, 1:11–60, 1996.

G. Shafer. *A mathematical theory of evidence*. Princeton University Press, 1976.

D. Tsementzis. A meaning explanation for HoTT. *Synthese*, Jan 2019. doi:10.1007/s11229-018-02052-1.

The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study, 2013. URL https://homotopytypetheory.org/book.

P. Walley. *Statistical Reasoning with Imprecise Probabilities*, volume 42 of *Monographs on Statistics & Applied Probability*. Chapman & Hall/CRC, 1990.

L. A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1:3–28, 1978.