Extensions of Sets of Markov Operators Under Epistemic Irrelevance

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Abstract

Sets of Markov operators can serve as generalised models for imprecise probabilities. They act on gambles as transformations preserving desirability. Often imprecise probabilistic models and also sets of operators need to be extended to larger domains. Such extensions are especially interesting when some kind of independence requirements have to be taken into account. The goal of this paper is to propose extension methods for sets of Markov operators that are consistent with the existing extension methods for imprecise probabilistic models. The main focus is on extensions satisfying epistemic irrelevance. We propose a new general approach to extending sets of desirable gambles, called *additive independent extension*, which subsumes important types of extensions, such as epistemic irrelevance and marginal extension. This approach is then extended to sets of Markov operators, so that the extensions are consistent with those of sets of desirable gambles.

Keywords: epistemic irrelevance, Markov operator, desirable gamble, additive independent extension

1. Introduction

Imprecise probabilistic models [1, 13] can be approached from three related ways. These are *coherent lower previsions, credal sets* and *sets of desirable gambles*. The benefit over classical models is that instead of having to specify the model precisely, the specification of the model can be partial and thus only represents the available information. The inferences obtained from the imprecise models are then considered more robust and reliable. Yet, in comparison with classical models, the specification of imprecise models is much more complex, in the best case it requires specification of convex sets of probabilities based on some constraints.

In [12] another even more general approach to modelling imprecise probabilities has been proposed. It uses sets of *Markov operators* that are in the role of *desirability preserving operators*. An example is *conditional expectation*, since a conditional expectation of a random gain is usually preferred to the original gain. Markov operators can also express other general judgements, such as symmetries between states. The main objective of this paper is finding extensions of sets of Markov operators to larger spaces that are consistent with independence concepts, especially epistemic irrelevance.

In the following two sections, the concepts of imprecise probabilities and the role of sets of Markov operators are briefly reviewed, with a special emphasis on their extensions. In Section 3.4 the concept of additive independent extension is proposed and then explored for the case of separately specified probabilistic models on a partition. In the last section, extensions of Markov operators are explored that lead to epistemically independent extensions of imprecise probabilistic models.

2. Imprecise Probabilities

2.1. Imprecise Probability Models

An imprecise probabilistic model denotes any probabilistic model which is not completely specified. This in general means that instead, a collection of judgements is given, to which possibly multiple probabilistic models correspond. The most natural way to describe an imprecise probabilistic model is therefore using a credal set, which is a set of precise or classical probabilistic models that correspond to given judgements. The formal introduction goes as follows. First, a sample space $\mathscr{X}$ is given together with an algebra $\mathscr{A}$ of its subsets. From now on, the set $\mathscr{X}$ will be finite and $\mathscr{A}$ will be its power set. Further, we consider the set of *gambles* $\mathscr{G}(\mathscr{X})$, which consists of all real valued maps $f\colon \mathscr{X} \to \mathbb{R}$. In the general case of possibly infinite $\mathscr{X}$, the gambles are additionally required to be $\mathscr{A}$-measurable. In our case, this requirement is automatically fulfilled.

A precise probabilistic model is now described in terms of a *linear prevision* P, which maps every gamble $f \in \mathscr{G}(\mathscr{X})$ into $P(f) = \sum_{x\in\mathscr{X}} p(x)f(x)$, where $p\colon \mathscr{X} \to \mathbb{R}$ is a *probability mass function* on $\mathscr{X}$. That is $p(x) \geq 0$ and $\sum_{x\in\mathscr{X}} p(x) = 1$. Now, a *credal set* $\mathscr{M}$ is just any closed and convex, thus compact, set of linear previsions. Once a credal set is given, it allows the construction of *coherent lower* and *upper prevision* pair $\underline{P}$ and $\overline{P}$:

$$\underline{P}(f) = \min_{P\in\mathscr{M}} P(f), \quad \overline{P}(f) = \max_{P\in\mathscr{M}} P(f). \tag{1}$$

The third equivalent way to introduce an imprecise probabilistic model is via a set of desirable gambles

$$\mathscr{D} = \{f \in \mathscr{G}(\mathscr{X})\colon P(f) \geq 0 \forall P \in \mathscr{M}\} \tag{2}$$

$$= \{f \in \mathscr{G}(\mathscr{X}) \colon \underline{P}(f) \geq 0\}. \tag{3}$$

Essentially, an imprecise probabilistic model can be described using any of the above models, yet there are some subtleties which make the three approaches not entirely equivalent ([10]). As in this paper we will be specifically concerned with the aspects of sets of desirable gambles, we additionally list their properties. A set $\mathscr{D}(\mathscr{X})$ is a subset of $\mathscr{G}(\mathscr{X})$ satisfying ([10]):

D1 if $f \geq 0$ then $f \in \mathscr{D}(\mathscr{X})$;

D2 if $f \in \mathscr{D}(\mathscr{X})$ and $\lambda > 0$, then $\lambda f \in \mathscr{D}(\mathscr{X})$;

D3 if $f, g \in \mathscr{D}(\mathscr{X})$ then $f + g \in \mathscr{D}(\mathscr{X})$;

D4 if $f \leq 0$ and $f \neq 0$ then $f \notin \mathscr{D}(\mathscr{X})$.

The above properties, however, are not universally accepted (see e.g [3, 4]). The main differences between different approaches concern borderline gambles, such as the constant zero gamble. To achieve equivalence between sets of desirable gambles and credal sets, which are assumed to be closed, closures of the sets of desirable gambles are often considered. The gambles in the closure are then referred as *almost desirable gambles*. For our approach it is beneficial to have closed sets of desirable gambles, and therefore we will slightly abuse the terminology by referring to sets of desirable gambles, even if they are in fact sets of almost desirable gambles. Thus, from now on, a set of desirable gambles denotes any closed set satisfying D1–D4. Moreover, to every set of desirable gambles $\mathscr{D}$, a credal set $\mathscr{M}$ and the corresponding lower prevision $\underline{P}$ can be associated, so that (2) and (3) hold.

Another convenient property of sets of desirable gambles is that, there always exists the obvious most conservative such set, that is the set of all non-negative gambles. This notion can be transferred to coherent lower previsions and credal sets, yet it is the most straightforward and intuitive in the case of gambles.

2.2. Extensions of Imprecise Probabilistic Models

Suppose that an imprecise probabilistic model is only defined on a subset $\mathscr{K}$ of gambles and the goal is to extend it to a larger set $\mathscr{G}$. An extension of the given model is an imprecise probabilistic model that coincides with the original model on $\mathscr{K}$. In general, multiple extensions exist, yet it is always possible to find the most conservative one, called the *natural extension*. The most conservative means that it allows the least gambles as desirable. Suppose that we have a set of desirable gambles $\mathscr{D}(\mathscr{K})$. Then its natural extension to $\mathscr{G}$ is the set $\mathscr{D}(\mathscr{G}) = \text{posi}\,(\mathscr{D}(\mathscr{K}) \cup \mathscr{G}_{\geq 0})$, where posi denotes the positive hull and $\mathscr{G}_{\geq 0}$ denotes the set of all non-negative gambles in $\mathscr{G}$.

In the sequel, the set $\mathscr{K}$ will most often be a (linear) subspace[1] of $\mathscr{G}$. In that case, the natural extension of a set of desirable gambles in $\mathscr{K}$ restricted to $\mathscr{K}$ equals the original set: $\mathscr{D}(\mathscr{G}) \cap \mathscr{K} = \text{posi}\,(\mathscr{D}(\mathscr{K}) \cup \mathscr{G}_{\geq 0}) \cap \mathscr{K} = \mathscr{D}(\mathscr{K})$. To see this, take some $f \in \mathscr{D}(\mathscr{G}) \cap \mathscr{K}$. We can write $f = f_{\mathscr{K}} + f_{\mathscr{G}}$, where $f_{\mathscr{K}} \in \mathscr{D}(\mathscr{K})$ and $f_{\mathscr{G}} \in \mathscr{G}_{\geq 0}$. By the fact that $\mathscr{K}$ is a subspace, we then have that $f_{\mathscr{G}} \in \mathscr{K}$ also holds, which implies that f is a sum of two desirable gambles in $\mathscr{K}$, which must belong to $\mathscr{D}(\mathscr{K})$. This proves that $\mathscr{D}(\mathscr{G}) \cap \mathscr{K} \subseteq \mathscr{D}(\mathscr{K})$. The reverse inclusion is an immediate consequence of the definition.

1. The term *subspace* will always denote a linear subspace.

Sets of desirable gambles on a subset $\mathscr{K}$ can be defined in two, in principle different ways. The first one is to have a closed set of gambles in $\mathscr{K}$ satisfying D1–D4, and the second one is to restrict a set of desirable gambles on the entire set $\mathscr{G}$ to $\mathscr{K}$. The above discussion implies that in the case where $\mathscr{K}$ is a subspace, both ways are equivalent.

Often, additional requirements are posed for extensions, such as independence or irrelevance of one variable with respect to another (see [5, 7, 9]). Let $\mathscr{X}, \mathscr{Y}$ and $\mathscr{Z}$ be three distinct sets. An imprecise probabilistic model on the space of gambles $\mathscr{G}(\mathscr{X} \times \mathscr{Y} \times \mathscr{Z})$ is said to satisfy *epistemic irrelevance* $\mathscr{Y} \to \mathscr{X}$ if learning the value of $\mathscr{Y}$ does not change our beliefs about values in $\mathscr{X}$. In [4] it has been shown that a set of desirable gambles $\mathscr{D}$ satisfies epistemic irrelevance $\mathscr{Y} \to \mathscr{X}$ exactly if for every $\mathscr{X}$-measurable f, the equivalence $f \in \mathscr{D} \Leftrightarrow I_y f \in \mathscr{D}$ for every $y \in \mathscr{Y}$ holds. The set $\mathscr{D}$ can thus be considered as an extension of $\mathscr{D}(\mathscr{X})$ to $\mathscr{G}(\mathscr{X} \times \mathscr{Y} \times \mathscr{Z})$.

Epistemic irrelevance denotes the situation where the unconditional imprecise probabilistic model on $\mathscr{X}$ is the same as the conditional models given the values $y \in \mathscr{Y}$. In [4], the conditional models are denoted by

$$\mathscr{D}\rfloor y = \{g \in \mathscr{G}(\mathscr{X} \times \mathscr{Z}) \colon I_y g \in \mathscr{D}\}. \tag{4}$$

In this paper, we will additionally define the subspace of gambles that is only non-zero for single values in $\mathscr{Y}$.

$$\mathscr{G}(\mathscr{X} \times \mathscr{Z}|y) = \{f \in \mathscr{G}(\mathscr{X} \times \mathscr{Y} \times \mathscr{Z}) \colon I_y f = f\} \tag{5}$$

and

$$\mathscr{D}(\mathscr{X} \times \mathscr{Z}|y) = \mathscr{D} \cap \mathscr{G}(\mathscr{X} \times \mathscr{Z}|y). \tag{6}$$

Similarly we can define $\mathscr{G}(\mathscr{X}|y)$ and $\mathscr{D}(\mathscr{X}|y)$ if the model does not contain a set $\mathscr{Z}$, or equivalently, if $\mathscr{Z}$ is a singleton. Similar constructions have also been used in [5].

3. Sets of Markov Operators

3.1. Consistency of Imprecise Probability Models with Sets of Markov Operators

We start with the definition.

Definition 1 *Let $\mathscr{H}$ and $\mathscr{K}$ be linear spaces of gambles on a set $\mathscr{X}$, containing all constant gambles.*

A linear operator $T \colon \mathscr{H} \to \mathscr{K}$ is called a Markov operator *or* stochastic operator *if* *(i) it is monotone: if $f \leq g$ then $Tf \leq Tg$;*

(ii) $T1_{\mathscr{X}} = 1_{\mathscr{X}}$.

In the current framework of finite sample spaces Markov operators can be represented in terms of stochastic matrices, i.e. positive matrices with row sums equal to ones. However, the notion of a Markov operator can be extended to infinite spaces as well. One such example are conditional expectations in general spaces considered in [11], which in fact motivated the use of Markov operators in the way presented here.

In [12] Markov operators are considered in the role of transformations preserving desirability, such as conditional expectation or permutation operators. The basic idea is very simple. Take, for instance, any desirable gamble f. Desirability is equivalent to the fact that the values $P(f)$ are non negative for every linear prevision P in some credal set, which can be interpreted as an operator mapping the gamble into a constant gamble. The credal set can then be interpreted as a set of expectation operators mapping general gambles to constant gambles: $P\colon f \to P(f)1_{\mathscr{X}}$. A more general set of Markov operators is obtained if conditional expectation operators are allowed. Such operators occur naturally in many cases, such as stochastic processes or risk modelling. Some interesting general results for this case have been proposed in [11, 12]. Another interesting case is where some kind of symmetry between states holds. Saying, for instance, that the available information about the likelihood of occurrence of x is the same as for y, can be modelled by requiring that for every gamble f which is desirable, so must be Tf, where $Tf(x) = f(y)$, $Tf(y) = f(x)$ and $Tf(z) = f(z)$ for every other element z in $\mathscr{X}$. It is easy to see that so defined operator T is a Markov operator.

In [12] three distinct notions of consistency of imprecise probabilistic models with Markov operators are proposed. All of them can be equivalently described using any of the forms of imprecise probabilistic models, credal sets, coherent lower prevision or sets of desirable gambles. Here we repeat the consistency notions for the sets of desirable gambles. Thus, let $\mathscr{D}(\mathscr{X})$ be a set of desirable gambles and $\mathscr{T}$ a closed set of Markov operators defined on the entire set of gambles $\mathscr{G}(\mathscr{X})$. (We will sometimes write $\mathscr{T}\colon \mathscr{H} \to \mathscr{K}$, to denote that every $T \in \mathscr{T}$ is a map $\mathscr{H} \to \mathscr{K}$.) Then $\mathscr{D}(\mathscr{X})$ is said to be consistent with $\mathscr{T}$ if $f \in \mathscr{D}(\mathscr{X})$ implies that $Tf \in \mathscr{D}(\mathscr{X})$ for every $T \in \mathscr{T}$. That is, every desirable gamble remains desirable after applying a desirability preserving operator to it. We will sometimes write

$$\mathscr{T}f = \{Tf \colon T \in \mathscr{T}\} \tag{7}$$

and

$$\mathscr{T}\mathscr{D} = \{Tf \colon T \in \mathscr{T}, f \in \mathscr{D}\} \tag{8}$$

and therefore, we can say that $\mathscr{D}$ is consistent with $\mathscr{T}$ if $\mathscr{T}\mathscr{D} \subseteq \mathscr{D}$.

The set $\mathscr{D}(\mathscr{X})$ is called dominated by $\mathscr{T}$ if for $f \in \mathscr{G}(\mathscr{X})$: $Tf \in \mathscr{D}(\mathscr{X})$ for every $T \in \mathscr{T}$ implies that $f \in \mathscr{D}(\mathscr{X})$. This is equivalent to saying that for every undesirable gamble, there is at least one operator that leaves it undesirable. If a set of desirable gambles $\mathscr{D}(\mathscr{X})$ is both consistent with and dominated by $\mathscr{T}$ then it is said to be generated by $\mathscr{T}$. This is the far most important notion because when applied to the special case of expectation operators, it corresponds to the usual relations between credal sets and the corresponding sets of desirable gambles.

A Markov operator T can also act from the right hand side on an expectation functional or linear prevision P, making PT again a linear prevision. Now we give the formal definition of the notion of imprecise probabilistic model, in any of the three forms, being generated by a set of Markov operators.

Definition 2 *Let $\mathscr{T}$ be a closed set of Markov operators defined on $\mathscr{G}$, $\mathscr{M}$ a credal set, $\underline{P}$ the corresponding lower prevision and $\mathscr{D}$ the corresponding set of desirable gambles. Then we say that the imprecise model, equivalently represented by $\mathscr{M}, \underline{P}$ or $\mathscr{D}$, is* generated by *$\mathscr{T}$ if the following equivalent conditions are fulfilled:*

(i) For every $f \in \mathscr{G} : f \in \mathscr{D}$ if and only if $Tf \in \mathscr{D}$ for every $T \in \mathscr{T}$.

(ii) For every linear prevision $P : P \in \mathscr{M}$ if and only if $PT \in \mathscr{M}$ for every $T \in \mathscr{T}$.

(iii) $\min_{T \in \mathscr{T}} \underline{P}(Tf) = \underline{P}f$ *for every gamble $f \in \mathscr{G}$.*

The proof for equivalence of the above conditions can be found in [12]. In general, multiple imprecise probabilistic models may be generated by the same set of Markov operators, which makes sets of Markov operators more flexible and more general than the ordinary imprecise probabilistic models. An evidence for this fact is given, for instance, in the Section 3.3 where it is shown that the natural extension and various marginal extensions are generated by the same set of Markov operators. This is not a coincidence, though, as it can be found in the above reference that the least committal generated model always exists.

3.2. Extension of Sets of Markov Operators

The problem that we analyse in this paper is how to extend sets of Markov operators defined on some particular subspaces of gambles. This question has already been partially addressed in [12]. Here we further investigate it, especially in the sense of how to extend sets of Markov operators to be in a meaningful relationship with extensions of the corresponding generated imprecise probabilistic models.

In [12] the following general theorem has been shown.

Theorem 3 *Let $\mathscr{K} \subset \mathscr{H}$ be linear spaces of gambles and $\mathscr{T}\colon \mathscr{K} \to \mathscr{K}$ a set of Markov operators and $\bar{\mathscr{T}}\colon \mathscr{H} \to \mathscr{H}$ its arbitrary extension. Then for every coherent lower prevision $\underline{P}$ on $\mathscr{K}$, generated by $\mathscr{T}$, there exists a coherent extension $\tilde{\underline{P}}$ to $\mathscr{H}$ that is generated by $\bar{\mathscr{T}}$.*

In the same paper, a version of the marginal extension theorem for sets of Markov operators has also been proved, and special attention has been given to the extensions to product spaces. Thus, if $\mathscr{T}$ and $\mathscr{S}$ are respectively sets of operators on the sets of gambles $\mathscr{G}(\mathscr{X})$ and $\mathscr{G}(\mathscr{Y})$, the extensions to $\mathscr{G}(\mathscr{X}\times\mathscr{Y})$ are studied whose marginals are generated by $\mathscr{T}$ and $\mathscr{S}$ respectively. In this direction, the results have been obtained for the case of the so-called strong independent product. The strong independent product of credal sets $\mathscr{M}$ and $\mathscr{N}$ corresponding to $\mathscr{G}(\mathscr{X})$ and $\mathscr{G}(\mathscr{Y})$ respectively is the credal set

$$\mathscr{M}\times\mathscr{N} = \{P\times Q\colon P\in\mathscr{M}, Q\in\mathscr{N}\}, \tag{9}$$

where $P\times Q$ denotes the product of linear previsions P and Q. It has been thus shown that every credal set of the form (9) satisfies the properties required above if it is generated by the tensor product

$$\mathscr{T}\otimes\mathscr{S} = \{T\otimes S\colon T\in\mathscr{T}, S\in\mathscr{S}\}. \tag{10}$$

In general it is not true that a set generated by a tensor product of sets of Markov operators would automatically be in the form of an independent product. The reason is that credal sets or sets of desirable gambles, which are just convex sets, have a much richer set of extensions than sets of linear operators.

In a similar way we will extend sets of operators in this paper, where the extensions will be in the consistency relations with imprecise probabilistic models, whose general form will be in a certain way pre-specified.

We will consider the problem of extension in a somewhat greater generality than Theorem 3. In general we will have a collection of linear subspaces $\mathscr{K}_i \subseteq \mathscr{G}$, where $\mathscr{G}$ is a linear space of gambles, and collections of sets of Markov operators $\mathscr{T}_i\colon \mathscr{K}_i \to \mathscr{K}_i$, for $i\in I$.

Definition 4 *A set of desirable gambles $\mathscr{D}\subseteq\mathscr{G}$ is generated by a set of Markov operators $\{\mathscr{T}_i\colon \mathscr{K}_i\to\mathscr{K}_i\}_{i\in I}$ if $\mathscr{D}\cap\mathscr{K}_i$ is generated by $\mathscr{T}_i$.*

Particularly, we are interested in sets of desirable gambles that are simultaneously generated by every set $\mathscr{T}_i$. This problem simplifies substantially if the operators $T_i\in\mathscr{T}_i$ are first in some meaningful way extended to a common domain $\mathscr{G}$. This is due to the following result.

Proposition 5 *Let $\{\mathscr{T}_i\colon \mathscr{G}\to\mathscr{G}\}_{i\in I}$ be a collection of sets of Markov operators. Then if $\mathscr{D}$ is generated by $\mathscr{T}_i$ for every $i\in I$, then $\mathscr{D}$ is generated by $\bigcup_{i\in I}\mathscr{T}_i$.*

Proof It is a direct consequence of the assumptions that if $\mathscr{D}$ is generated by $\mathscr{T}_i$ for every i, then for every $f\in\mathscr{G}$ we have that $f\in\mathscr{D}$ if and only if $T_i f\in\mathscr{D}$ for every $T_i\in\mathscr{T}_i$ and every $i\in I$, which is equivalent to $Tf\in\mathscr{D}$ for every $T\in\bigcup_{i\in I}\mathscr{T}_i$, which in turn is equivalent to $\mathscr{D}$ being generated by $\bigcup_{i\in I}\mathscr{T}_i$. ■

The reverse of the above proposition is clearly not true. That is, if a set $\mathscr{D}$ happens to be generated by $\bigcup_{i\in I}\mathscr{T}_i$, then it is not necessarily generated by each individual $\mathscr{T}_i$.

3.3. Separately Specified Conditional Models

We will first consider the case where the probabilistic models to be extended are specified on separate probability spaces. Let $\mathscr{B}$ be a partition of $\mathscr{X}$ and $B\in\mathscr{B}$. Then $\mathscr{G}(B)$ will denote the set of gambles with support B, i.e. $f\in\mathscr{G}(B)$ if and only if $I_B f = f$. Further let $\mathscr{G}(\mathscr{B})$ denote the set of $\mathscr{B}$-measurable gambles, i.e. gambles that are constant on every $B\in\mathscr{B}$.

Let $\mathscr{T}_B\colon \mathscr{G}(B)\to\mathscr{G}(B)$ be a set of Markov operators. We call $\mathscr{T}_B$ a *conditional model* on B. Let $\mathscr{D}(B)$ be a set of desirable gambles with support B that is generated by $\mathscr{T}_B$. (We will always assume that $\mathscr{D}(B)$ only contains gambles with support B.) An *extension* of $\mathscr{D}(B)$ is any set of desirable gambles $\mathscr{D}$ such that $\mathscr{D}\cap\mathscr{G}(B)=\mathscr{D}(B)$. By Definition 4, an extension of $\mathscr{D}(B)$ is generated by $\mathscr{T}_B$.

The smallest set of desirable gambles on $\mathscr{G}(\mathscr{X})$ that extends every $\mathscr{D}(B)$ is well known and is called the *natural extension*:

$$\mathscr{D}_n = \left\{\sum_{B\in\mathscr{B}} f_B\colon f_B\in\mathscr{D}(B)\right\}. \tag{11}$$

The natural extension is clearly the minimal extension with the required properties, which follows from the fact that it contains exactly the sums of desirable gambles, which by basic properties must be contained in a set of desirable gambles.

Yet, the natural extension is not the only possible extension of the local models to the entire space of gambles. Actually, it is only a special case of the *marginal extension* ([9]), which is obtained as follows. Let $\mathscr{D}(B)$ be a set of desirable gambles for every $B\in\mathscr{B}$; and $\mathscr{D}(\mathscr{B})$ a set of $\mathscr{B}$-measurable desirable gambles. Then the marginal extension of these models is the set

$$\mathscr{D}_m = \left\{f_{\mathscr{B}} + \sum_{B\in\mathscr{B}} f_B\colon f_{\mathscr{B}}\in\mathscr{D}(\mathscr{B}), f_B\in\mathscr{D}(B)\,\forall B\in\mathscr{B}\right\} \tag{12}$$

$$= \mathscr{D}(\mathscr{B}) + \mathscr{D}_n. \tag{13}$$

The above construction can be found, for instance, in [10], Section 2.3. The natural extension is therefore the marginal extension where $\mathscr{D}(\mathscr{B})$ is the set of non-negative gambles, which is the minimal such set of desirable gambles.

Being able to extend sets of desirable gamble generated by sets of operators from subspaces of gambles to the entire space of gambles $\mathscr{G}(\mathscr{X})$, we now extend the sets of operators themselves in such a way to be related with the corresponding extensions of the sets of desirable gambles.

Given a collection of separately specified conditional models given by sets of Markov operators, an obvious way of extending them to the entire set $\mathscr{G}(\mathscr{X})$ is the following.

Definition 6 *Let $\mathscr{B}$ be a partition and $\{\mathscr{T}_B : \mathscr{G}(B) \to \mathscr{G}(B)\}_{B\in\mathscr{B}}$ a collection of sets of Markov operators. Then we define the set*

$$\tilde{\mathscr{T}} = \left\{ \tilde{T} : \tilde{T}f = \sum_{B\in\mathscr{B}} T_B(I_B f), T_B \in \mathscr{T}_B \forall B \in \mathscr{B} \right\}. \quad (14)$$

Proposition 7 *Every $\tilde{T} \in \tilde{\mathscr{T}}$ is a Markov operator on $\mathscr{G}(\mathscr{X})$, and $\tilde{\mathscr{T}}|_{\mathscr{G}(B)} = \mathscr{T}_B$ for every $B \in \mathscr{B}$.*

Restricted to $\mathscr{G}(B)$, the sets of desirable gambles generated by $\mathscr{T}_B$ and $\tilde{\mathscr{T}}$ clearly coincide, which however does not imply that the sets of desirable gambles for the entire $\mathscr{G}(\mathscr{X})$ necessarily also coincide. Yet they do coincide in an important special case described by the following proposition.

Proposition 8 *Let $\mathscr{D}$ be a marginal extension of a collection of local sets of desirable gambles $\{\mathscr{D}(B)\}_{B\in\mathscr{B}}$ and some set of $\mathscr{B}$-measurable desirable gambles $\mathscr{D}(\mathscr{B})$ (see (13)). Then $\mathscr{D}(B)$ is generated by $\mathscr{T}_B$ for every $B \in \mathscr{B}$ if and only if $\mathscr{D}$ is generated by $\tilde{\mathscr{T}}$.*

We will postpone the proof to the next section.

The set $\tilde{\mathscr{T}}$ thus generates the same marginal extensions as the corresponding local models. We can therefore regard it as an extension of the local models. The family of all extensions of the local models is of course much richer than the family of marginal extensions alone. Yet, in the next section we will show that not only is every marginal extension generated by the extended sets of operators, but also that exactly the marginal extensions satisfy a new minimality property of extended sets of desirable gambles.

3.4. Additive Independent Extension

We now define a general type of extensions of local imprecise probabilistic models to a global one which turns out to be consistent with extensions of sets of Markov operators in Definition 6.

Definition 9 *Let $\mathscr{D}$ be a set of desirable gambles that extends sets of desirable gambles $\mathscr{D}_i \subseteq \mathscr{G}_i$ for $i \in I$, where $\mathscr{G}_i$ are subspaces of the set of all gambles $\mathscr{G}$ and I is a finite set of indices. Then we will say that $\mathscr{D}$ is an* additive independent extension (AIE) *of $\{\mathscr{D}_i\}_{i\in I}$ if for every $f \in \mathscr{D}$ that can be written as a sum $f = \sum_{i\in I} f_i$, where $f_i \in \mathscr{G}_i$, implies that $f_i \in \mathscr{D}_i$ for at least one $i \in I$.*

In other words, $\mathscr{D}$ is an additive independent extension of $\{\mathscr{D}_i\}_{i\in I}$ if a sum of undesirable gambles from different $\mathscr{G}_i$ is always undesirable. The main motivation for the above definition comes from the situation explained in Example 1.

Proposition 10 *The natural extension (11) is an AIE of $\{\mathscr{D}(B)\}_{B\in\mathscr{B}}$.*

The above proposition is a special case of Theorem 13 below.

An AIE does not always exist. The following proposition tells us that such an extension is only possible if the probabilistic models are linear on the intersections, which is equivalent to being the maximal coherent sets of desirable gambles. These have been defined in [4, 5], as sets of desirable gambles, to which no further gambles can be added. More precisely, a coherent set of desirable gambles $\mathscr{D}$ is maximal if $f \notin \mathscr{D}$ implies that $-f \in \mathscr{D}$ for every gamble $f \in \mathscr{G}$[2].

Proposition 11 *Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be subspaces of gambles on $\mathscr{X}$. Let $\mathscr{D}$ be an AIE of $\mathscr{D}_1 \subseteq \mathscr{K}_1$ and $\mathscr{D}_2 \subseteq \mathscr{K}_2$. Then $\mathscr{D} \cap \mathscr{K}_1 \cap \mathscr{K}_2$ is a maximal coherent set of desirable gambles in the subspace $\mathscr{K}_1 \cap \mathscr{K}_2$.*

Proof First notice that $\mathscr{D}_1 \cap \mathscr{K}_1 \cap \mathscr{K}_2 = \mathscr{D}_2 \cap \mathscr{K}_1 \cap \mathscr{K}_2 = \mathscr{D} \cap \mathscr{K}_1 \cap \mathscr{K}_2$, because otherwise $\mathscr{D}$ could not be considered an extension of $\mathscr{D}_1$ and $\mathscr{D}_2$.

Take some $f \in \mathscr{K}_1 \cap \mathscr{K}_2$. Then $f + (-f) = 0 \in \mathscr{D}$. Whence either f or $-f$ belongs to $\mathscr{D}_1 \cap \mathscr{K}_1 \cap \mathscr{K}_2 = \mathscr{D}_2 \cap \mathscr{K}_1 \cap \mathscr{K}_2 \subseteq \mathscr{D}_1 \cap \mathscr{D}_2$, which by the above discussion implies that $\mathscr{D}_1 \cap \mathscr{D}_2$ is a maximal coherent set. ∎

The following proposition gives a sufficient condition for the existence of an AIE.

Proposition 12 *Let $\{\mathscr{K}_i\}_{i\in I}$ be a finite collection of linear spaces of gambles $\mathscr{K}_i \subseteq \mathscr{G}(\mathscr{X})$ for every $i \in I$, such that $\mathscr{K}_i \cap \sum_{j\neq i} \mathscr{K}_j \subseteq$ Const, where Const denotes the set of constant gambles. Further let $\{\mathscr{D}_i\}_{i\in I}$ denote a corresponding collection of sets of desirable gambles: $\mathscr{D}_i \subset \mathscr{K}_i$. Then $\mathscr{D} = \{\sum_{i\in I} f_i : f_i \in \mathscr{D}_i \forall i \in I\}$ is an AIE.*

Proof To see that $\mathscr{D}$ is an AIE, take any sum $\sum_{i\in I} g_i \in \mathscr{D}$. We need to show that at least one g_i is desirable. By the construction, there exists a collection of $f_i \in \mathscr{D}_i$ for every $i \in I$ such that $\sum_{i\in I} g_i = \sum_{i\in I} f_i$. This implies that

$$\sum_{i\in I} (g_i - f_i) = 0, \quad (15)$$

and consequently, for every i, the equality $g_i - f_i = \sum_{j\neq i}(f_j - g_j)$ holds. The left hand side of the above equality lies in $\mathscr{K}_i$, while the right hand side is in the sum of the remaining spaces. This, by assumption, implies that both

2. The original definition requires $f \neq 0$, which is not needed in our case where 0 is deemed desirable.

sides must be constants. Thus (15) is a sum of constants adding to 0, and therefore, at least one of the constants $g_i - f_i$ must be non-negative, and for such i this implies that $g_i \geq f_i$, and since $f_i \in \mathscr{D}_i$, g_i must be desirable as well, which completes the proof that $\mathscr{D}$ is an AIE. ∎

A full characterization of AIE for separately specified conditional models follows.

Theorem 13 *Let a collection $\{\mathscr{D}(B)\}_{B\in\mathscr{B}}$ of sets of desirable gambles and $\mathscr{D}$ a set of desirable gambles in $\mathscr{G}(\mathscr{X})$ be given. Then $\mathscr{D}$ is an AIE if and only if there exists a set of $\mathscr{B}$-measurable desirable gambles $\mathscr{D}(\mathscr{B})$ such that for every $f \in \mathscr{D}, f_{\mathscr{B}} \in \mathscr{D}(\mathscr{B})$ and $f_B \in \mathscr{D}_B$ for each $B \in \mathscr{B}$ exist, so that*

$$f = f_{\mathscr{B}} + \sum_{B\in\mathscr{B}} f_B. \tag{16}$$

In other words, any AIE of $\{\mathscr{D}(B)\}_{B\in\mathscr{B}}$ is of the form $\mathscr{D} = \mathscr{D}(\mathscr{B}) + \sum_{B\in\mathscr{B}} \mathscr{D}(B)$ for some $\mathscr{D}(\mathscr{B})$.

Proof Let us first show the 'if' part. Thus, assume that for every $f \in \mathscr{D}$, decomposition (16) exists. We need to show that in this case, some $B \in \mathscr{B}$ exists so that $I_B f \in \mathscr{D}(B)$. Since $f_{\mathscr{B}}$ is assumed to be desirable, there must be some $B \in \mathscr{B}$, such that $f_{\mathscr{B}}(B) \geq 0$. This implies that $1_B f = 1_B f_{\mathscr{B}}(B) + f_B$, where both terms are clearly desirable, and therefore their sum is desirable as well.

Now take some $f \in \mathscr{D}$, and let $\lambda_B = \max\{\lambda : I_B f - \lambda \in \mathscr{D}(B)\}$. Set $f_{\mathscr{B}}(B) = \lambda_B$ and $f_B = I_B(f - \lambda_B)$ for every $B \in \mathscr{B}$. We then have that $f = f_{\mathscr{B}} + \sum_{B\in\mathscr{B}} f_B$.

Consider the set $\mathscr{D}(\mathscr{B}) = \{f_{\mathscr{B}} : f \in \mathscr{D}\}$. We show that it is a set of weakly desirable gambles. Clearly it contains all non-negative gambles, which is equivalent to $\mathscr{D}$ containing the minimal extension.

Further, if $f_{\mathscr{B}} \in \mathscr{D}(\mathscr{B})$ then it must hold that $f_{\mathscr{B}}(B) \geq 0$ for at least one $B \in \mathscr{B}$. Indeed, if $f_{\mathscr{B}}(B) < 0$ for all $B \in \mathscr{B}$, then for every B we have that $I_B f = I_B(I_B f - f_{\mathscr{B}}(B) + f_{\mathscr{B}}(B))$. But as $f_{\mathscr{B}}(B)$ is the maximal constant that can be subtracted from $I_B f$ to remain desirable, $I_B f$ therefore cannot be desirable. This contradicts with $\mathscr{D}$ being an additive independent extension, and therefore proves that $f_{\mathscr{B}}(B) \geq 0$ for some $B \in \mathscr{B}$.

Finally we need to prove that $\mathscr{D}(\mathscr{B})$ is a convex set. Take $f, g \in \mathscr{D}$ and the corresponding $f_{\mathscr{B}}$ and $g_{\mathscr{B}}$. Since for every $B \in \mathscr{B}$, $I_B(f - f_{\mathscr{B}}(B))$ and $I_B(g - g_{\mathscr{B}}(B))$ are both desirable, that is, belong to $\mathscr{D}(B)$, so must be their sum $I_B(f + g - (f_{\mathscr{B}}(B) + g_{\mathscr{B}}(B)))$. By definition then $(f + g)_{\mathscr{B}}(B) \geq f_{\mathscr{B}}(B) + g_{\mathscr{B}}(B)$, and since $(f+g)_{\mathscr{B}} \in \mathscr{D}(\mathscr{B})$, so must $f_{\mathscr{B}} + g_{\mathscr{B}} \in \mathscr{D}(\mathscr{B})$.

It is immediate that $(\alpha f)_{\mathscr{B}} = \alpha f_{\mathscr{B}}$ for $\alpha \geq 0$, which together with the above proves convexity of $\mathscr{D}(\mathscr{B})$. Moreover, it is clear that the map $f \mapsto f_{\mathscr{B}}$ is continuous and therefore maps the closed set $\mathscr{D}$ to a closed set $\mathscr{D}(\mathscr{B})$, which makes it a set of desirable gambles according to our definition. ∎

Very similarly we can show the following.

Proposition 14 *Let under the assumptions of Theorem 13, $g \in \mathscr{G}(\mathscr{X})$ be given. Then $g \notin \mathscr{D}$ if and only if there exist $g'_{\mathscr{B}}$ and g'_B for every $B \in \mathscr{B}$ such that neither of them are desirable and*

$$g = g'_{\mathscr{B}} + \sum_{B\in\mathscr{B}} g'_B. \tag{17}$$

Proof First we take some $g \notin \mathscr{D}$ and show that the required decomposition exists. The construction will be similar to the one in the proof of Theorem 13. In particular, g_B are constructed in the same way as f_B before. Now, since all g_B are (marginally) desirable, g can only be undesirable if $g_{\mathscr{B}}$ is undesirable. By our definition, the set of undesirable gambles is open, which means that $g'_{\mathscr{B}} = g_{\mathscr{B}} + \varepsilon$ is undesirable for some $\varepsilon > 0$. Now taking $g'_B = g_B - I_B\varepsilon$ makes g'_B undesirable, and since $g = g'_{\mathscr{B}} + \sum_{B\in\mathscr{B}} g'_B$, this proves our proposition.

For the reverse implication, suppose that g is of the form (17). Assume ex-absurdo that $g \in \mathscr{D}$, in which case by Theorem 13, a decomposition $g = f_{\mathscr{B}} + \sum_{B\in\mathscr{B}} f_B$ exists, where all the terms are desirable. This implies that

$$g'_{\mathscr{B}} - f_{\mathscr{B}} = \sum_{B\in\mathscr{B}} f_B - g'_B. \tag{18}$$

Since the gamble on the left hand side is $\mathscr{B}$-measurable, all the differences $f_B - g'_B$ must be constant on the corresponding B, and since all f_B are desirable and g'_B are not, they must be positive constants. This would imply that $g'_{\mathscr{B}} \geq f_{\mathscr{B}}$, and therefore, if $f_{\mathscr{B}}$ is desirable, so must be $g'_{\mathscr{B}}$. But this contradicts our initial requirement, and therefore completes this part of the proof. ∎

Theorem 15 *Let $\mathscr{T}_B : \mathscr{G}(B) \to \mathscr{G}(B)$ be a collection of conditional models for each $B \in \mathscr{B}$, where $\mathscr{B}$ is a partition of $\mathscr{X}$. Let $\mathscr{D}$ be a set of desirable gambles that it is an AIE of $\{\mathscr{D}(B) = \mathscr{D} \cap \mathscr{G}(B)\}_{B\in\mathscr{B}}$. Then each $\mathscr{D}(B)$ is generated by $\mathscr{T}_B$ if and only if $\mathscr{D}$ is generated by $\tilde{\mathscr{T}}$, from Definition 6.*

Proof Suppose first that $\mathscr{D}$ is generated by $\tilde{\mathscr{T}}$. Now we show that $\mathscr{D} \cap \mathscr{G}(B) = \mathscr{D}(B)$ is generated by $\mathscr{T}_B$. Take some $f \in \mathscr{G}(B)$. Then we have that $f \in \mathscr{D}$ if and only if $\tilde{\mathscr{T}} f \subseteq \mathscr{D}$. By Proposition 7, $\tilde{\mathscr{T}} f = \mathscr{T}_B f$, and we have that $f \in \mathscr{D} \cap \mathscr{G}(B) = \mathscr{D}(B)$ if and only if $\tilde{\mathscr{T}} f = \mathscr{T}_B f \subset \mathscr{D}$ and since $\mathscr{T}_B$ maps f to $\mathscr{G}(B)$, we further have that $\mathscr{T}_B f \subset \mathscr{D}(B)$.

Now we prove the reverse implication. Thus suppose that $\mathscr{D}$ is an AIE of $\mathscr{D}(B)$, where every $\mathscr{D}(B)$ is generated by $\mathscr{T}_B$. We have to show that $\mathscr{D}$ is then generated by $\tilde{\mathscr{T}}$. Thus we need to show that $f \in \mathscr{D}$ holds if and only if $\tilde{T} f \in \mathscr{D}$ for every $\tilde{T} \in \tilde{\mathscr{T}}$. Take some $f \in \mathscr{D}$ first. By Theorem 13, it is of the form $f = f_{\mathscr{B}} + \sum_{B\in\mathscr{B}} f_B$, where $f_{\mathscr{B}}$ belongs to some set of $\mathscr{B}$-measurable desirable gambles and $f_B \in \mathscr{D}(B)$ for every $B \in \mathscr{B}$. Now for arbitrary $\tilde{T} \in \tilde{\mathscr{T}}$ we then have that $\tilde{T} f = f_{\mathscr{B}} + \sum_{B\in\mathscr{B}} T_B f_B$, because $f_{\mathscr{B}}$ is constant on every $B \in \mathscr{B}$. Since $\mathscr{D}(B)$ are all generated by $\mathscr{T}_B$, the inclusions

$T_B f_B \in \mathscr{D}(B)$ hold for every $B \in \mathscr{B}$, and therefore $\tilde{T} f \in \mathscr{D}$ holds as well.

For the second part, take some $g \notin \mathscr{D}$. By Proposition 14, we then have that $g = g'_{\mathscr{B}} + \sum_{B\in\mathscr{B}} g'_B$, where neither of the terms in the sum are desirable. By the definition of a generated set, for every $g_B \notin \mathscr{D}(B)$ there must exist some $T_B \in \mathscr{T}_B$, so that $T_B g_B \notin \mathscr{D}(B)$. Now taking $\tilde{T}$ to be the operator such that $\tilde{T}|_{\mathscr{G}(B)} = T_B$, we have that $\tilde{T} g$ is still a sum of non-desirable gambles, which again by Proposition 14 implies that $\tilde{T} g \notin \mathscr{D}$. ■

In Theorem 15 we only extend conditional imprecise probabilistic models to a model that is given up to the marginal $\mathscr{B}$-measurable model. This demonstrates the advantage of sets of operators over imprecise probabilistic models, as the former may be consistent with multiple imprecise probabilistic models. In our case, the lack of marginal information still allows us to build a global set of operators and construct models consistent with it.

The following proposition now shows how easy it is to combine the extended local models with the marginal model.

Proposition 16 *Let $\mathscr{B}$ be a partition of $\mathscr{X}$, $\mathscr{T}_B$ conditional model for every $B \in \mathscr{B}$ and $\mathscr{T}_{\mathscr{B}}$ a set of Markov operators on the space of $\mathscr{B}$-measurable gambles. Let $\mathscr{D}(\mathscr{B})$ and $\mathscr{D}(B)$ be the corresponding generated sets of desirable gambles. Then the set*

$$\mathscr{D} = \mathscr{D}(\mathscr{B}) + \sum_{B\in\mathscr{B}} \mathscr{D}(B) \tag{19}$$

is their AIE.

Proof Clearly $\mathscr{D}$ is a set of desirable gambles. We now show that it is an AIE of the corresponding sets. Thus take some gamble $f = g_{\mathscr{B}} + \sum_{B\in\mathscr{B}} g_B$ where $g_{\mathscr{B}} \in \mathscr{G}(\mathscr{B}), g_B \in \mathscr{G}(B)$ for all $B \in \mathscr{B}$, so that $f \in \mathscr{D}$. We must show that at least one of the gambles in the sum is desirable. Since $f \in \mathscr{D}$, it can be written in the form $f = f_{\mathscr{B}} + \sum_{B\in\mathscr{B}} f_B$, where all of the gambles in the sum are desirable. Now we have that $f_{\mathscr{B}} - g_{\mathscr{B}} = \sum_{B\in\mathscr{B}} g_B - f_B$. The left hand side of the above relation is $\mathscr{B}$-measurable, and therefore all $g_B - f_B$ are constant on B. Now suppose that $g_{\mathscr{B}}$ is not desirable. In that case $f_{\mathscr{B}} - g_{\mathscr{B}}$ must be non-negative on some $B \in \mathscr{B}$, whence $g_B - f_B \geq 0$ and therefore $g_B \geq f_B$ and since f_B is desirable, so must be g_B. This completes the proof. ■

Definition 17 *Let $\mathscr{B}$ be a partition of $\mathscr{X}$ and $\mathscr{T}_B \colon \mathscr{G}(B) \to \mathscr{G}(B)$ sets of Markov operators. Then we call the set $\tilde{\mathscr{T}}$ from Definition 6 an* additive independent extension *of $\{\mathscr{T}_B\}_{B\in\mathscr{B}}$.*

Definition 18 *Let $\mathscr{T}_i \colon \mathscr{G}_i \to \mathscr{G}_i$ for $i \in I$ be sets of Markov operators, where $\mathscr{G}_i$ are some subspaces of gambles for some index set I. Let $\mathscr{D}$ be a set of desirable gambles that is an AIE of $\{\mathscr{D} \cap \mathscr{G}_i\}_{i\in I}$, where each $\mathscr{D} \cap \mathscr{G}_i$ is generated by $\mathscr{T}_i$. Then we will say that $\mathscr{D}$ is an* additive independent extension *of $\{\mathscr{T}_i\}_{i\in I}$.*

4. Epistemic Irrelevance

In this section we show how AIE fits into the concept of epistemic irrelevance and how extensions of local probabilistic models given in terms of sets of Markov operators can be used to construct global probabilistic models that satisfy epistemic irrelevance. An area where such extensions have been used intensively is the theory of imprecise stochastic processes [8], and in particular imprecise Markov chains. De Cooman et al. [6], whose paper has been used as the basic framework for most of the research activities in the field of imprecise stochastic processes, build a global probabilistic model from local one-step transition models under the assumption of epistemic irrelevance.

4.1. Epistemic Irrelevance Between Two Variables

First we consider two finite sets $\mathscr{X}$ and $\mathscr{Y}$ and their Cartesian product $\mathscr{X} \times \mathscr{Y}$. We will denote the gambles on the corresponding spaces with $\mathscr{G}(\mathscr{X}), \mathscr{G}(\mathscr{Y})$ and $\mathscr{G}(\mathscr{X} \times \mathscr{Y})$. A set of desirable gambles $\mathscr{D} \subseteq \mathscr{G}(\mathscr{X} \times \mathscr{Y})$ is said to satisfy epistemic irrelevance $\mathscr{Y} \to \mathscr{X}$ if every $\mathscr{X}$-measurable gamble f belongs to $\mathscr{D}$ if and only if $I_y f$ belongs to $\mathscr{D}$ for every $y \in \mathscr{Y}$ ([4]). This implies that all conditional models are the same regardless of $y \in \mathscr{Y}$. Using sets of Markov operators we can describe this property as follows. Let σ be a permutation on $\mathscr{Y}$. Then we define the *permutation operator*

$$[P^\sigma f](x,y) = f(x,\sigma(y)). \tag{20}$$

As the idea of epistemic irrelevance is that learning y does not change our beliefs about $\mathscr{X}$, it would be reasonable to expect that any gamble f is desirable exactly if $P^\sigma f$ is. We will then say that the model satisfies *symmetry* with respect to $\mathscr{Y}$. We will denote the set of permutation operators with respect to $\mathscr{Y}$ with $\mathscr{P}(\mathscr{Y})$.

Let a set of desirable gambles $\mathscr{D} \subset \mathscr{G}(\mathscr{X} \times \mathscr{Y})$ be given. Then for every $y \in \mathscr{Y}$ we define $\mathscr{D}_y = \{f \in \mathscr{G}(\mathscr{X}) \colon I_y f \in \mathscr{D}\} \subseteq \mathscr{D}(\mathscr{X})$. The following proposition is immediate.

Proposition 19 *Let $\mathscr{D} \subset \mathscr{G}(\mathscr{X} \times \mathscr{Y})$ be a set of desirable gambles that is generated by $\mathscr{P}(\mathscr{Y})$. Then $\mathscr{D}_y = \mathscr{D}_{y'}$ for every pair $y, y' \in \mathscr{Y}$.*

Proof Take some $f \in \mathscr{G}(\mathscr{X})$. Then $f \in \mathscr{D}_y$ is equivalent to $I_y f \in \mathscr{D}$. Now since $\mathscr{D}$ is generated by $\mathscr{P}(\mathscr{Y})$, this implies that $[P^\sigma I_y f] = I_{\sigma(y)} f \in \mathscr{D}$, whence $f \in \mathscr{D}_{\sigma(y)}$. Now since $\sigma(y)$ takes all values of $\mathscr{Y}$ this implies that all $\mathscr{D}_y$ are the same for every $y \in \mathscr{Y}$. ■

Symmetry however is not enough to guarantee epistemic irrelevance. To see this, consider the following example.

Example 1 *Consider the case where* $\mathscr{X} = \mathscr{Y} = \{0,1\}$ *and let* $f(0,y) = -1$ *and* $f(1,y) = 1$ *for* $y = 0,1$. *Further take two probability mass functions:*

$$\begin{aligned} p_1(0,0) &= 3/16 & p_1(0,1) &= 3/16, \\ p_1(1,0) &= 2/16 & p_1(1,1) &= 8/16; \\ p_2(0,0) &= 3/16 & p_2(0,1) &= 3/16, \\ p_2(1,0) &= 8/16 & p_2(1,1) &= 2/16. \end{aligned}$$

Let $\mathscr{M} = \{p_1, p_2\}$ *and the corresponding lower prevision* $\underline{P}(h) = \min_{p\in\mathscr{M}} E_p(h)$. *We then obtain*

$$\underline{P}(f) = 4/16, \underline{P}(I_0 f) = -1/16, \underline{P}(I_1 f) = -1/16,$$

which shows that although both $I_0 f$ *and* $I_1 f$ *are not desirable,* f*, their sum, is a desirable gamble. Thus, despite clear symmetry with respect to* $\mathscr{Y}$*, the corresponding imprecise probabilistic model does not satisfy epistemic irrelevance.*

The reason that the imprecise probabilistic model in the above example does not satisfy epistemic irrelevance is in the fact that the sum of undesirable gambles $I_{\{y=0\}} f$ and $I_{\{y=1\}} f$ is desirable. Clearly such a situation can be prevented if the set of desirable gambles is an additive independent extension.

Let the conditional models $\mathscr{D}(\mathscr{X}|y)$ be generated by sets of Markov operators $\mathscr{T}_y$ on $\mathscr{G}(\mathscr{X}|y)$. Since all $\mathscr{G}(\mathscr{X}|y)$ are isomorphic with $\mathscr{G}(\mathscr{X})$, we can additionally require that all $\mathscr{T}_y$ are equal for every $y \in \mathscr{Y}$. This does not automatically imply that the conditional models are equal as well, because there may be multiple sets of desirable gambles generated by the same set of Markov operators. Thus equality of the models will be guaranteed by employing the permutation operators.

Theorem 20 *Let* $\mathscr{D} \subset \mathscr{G}(\mathscr{X} \times \mathscr{Y})$ *be a set of desirable gambles that is an AIE of* $\{\mathscr{D}(\mathscr{X}|y)\}_{y\in\mathscr{Y}}$*, which are generated by the corresponding* $\mathscr{T}_y$*, where* $\mathscr{T}_y = \mathscr{T}_\mathscr{X}$ *for every* $y \in \mathscr{Y}$. *Further let* $\mathscr{D}$ *be additionally generated by* $\mathscr{P}(\mathscr{Y})$. *Then it satisfies epistemic irrelevance* $\mathscr{Y} \to \mathscr{X}$.

Proof We need to prove that for every $f \in \mathscr{G}(\mathscr{X})$ and $y \in \mathscr{Y}, f \in \mathscr{D}$ if and only if $I_y f \in \mathscr{D}$ for all $y \in \mathscr{Y}$. The 'if' part is immediate, because $I_y f \in \mathscr{D}$ for every $y \in \mathscr{Y}$ implies that their sum $\sum_{y\in\mathscr{Y}} I_y f$, which equals f, is desirable as well.

Thus, assume now that $f \in \mathscr{D}$, and as it is the sum of $I_y f$, which respectively belong to $\mathscr{G}(\mathscr{X}|y)$, at least one of them, say $I_{y_0} f$ is desirable. Further, as $\mathscr{D}$ is also generated by $\mathscr{P}(\mathscr{Y})$, we have that $P^\sigma[I_{y_0} f] = I_{\sigma(y_0)} f$ must be desirable as well. But since this holds for every permutation $\sigma, \sigma(y_0)$ can assume any element of $\mathscr{Y}$, implying that $I_y f$ is desirable for every $y \in \mathscr{Y}$. ■

Corollary 21 *Let* $\mathscr{D}$ *be a set of desirable gambles satisfying the conditions of Theorem 20. Then there exist a set of desirable gambles* $\mathscr{D}(\mathscr{Y}) \subseteq \mathscr{G}(\mathscr{Y})$ *that is generated by* $\mathscr{P}(\mathscr{Y})$ *and a set of desirable gambles* $\mathscr{D}(\mathscr{X})$ *so that*

$$\mathscr{D} = \left\{ f_\mathscr{Y} + \sum_{y\in\mathscr{Y}} I_y f_y \colon f_\mathscr{Y} \in \mathscr{D}(\mathscr{Y}), f_y \in \mathscr{D}(\mathscr{X}) \forall y \in \mathscr{Y} \right\} \tag{21}$$

$$= \mathscr{D}(\mathscr{Y}) + \sum_{y\in\mathscr{Y}} I_y \mathscr{D}(\mathscr{X}). \tag{22}$$

Proof By Theorem 13 there exist some $\mathscr{D}(\mathscr{Y})$ and $\mathscr{D}_y \subseteq \mathscr{G}(\mathscr{X})$ for each $y \in \mathscr{Y}$ so that every $f \in \mathscr{D}$ can be written in the form $f = f_\mathscr{Y} + \sum_{y\in\mathscr{Y}} I_y f_y$, where $f_\mathscr{Y} \in \mathscr{D}(\mathscr{Y})$ and $f_y \in \mathscr{D}_y$.

Each individual $\mathscr{D}_y$ is equal to $\mathscr{D}(\mathscr{X})$, which is required by epistemic irrelevance. Now take some $f_\mathscr{Y} + \sum_{y\in\mathscr{Y}} I_y f_y$. Since $\mathscr{D}$ is generated by $\mathscr{P}(\mathscr{Y})$, we have that $P^\sigma f = P^\sigma f_\mathscr{Y} + \sum_{y\in\mathscr{Y}} I_{\sigma(y)} f_y$ is desirable as well. This is only possible if $P^\sigma f_\mathscr{Y}$ belongs to $\mathscr{D}(\mathscr{Y})$ as well. ■

Definition 22 *Let* $\mathscr{T}_\mathscr{X} \colon \mathscr{G}(\mathscr{X}) \to \mathscr{G}(\mathscr{X})$ *be a set of Markov operators and let* $\mathscr{Y}$ *be another sample space. Then the extension* $\tilde{\mathscr{T}} \colon \mathscr{G}(\mathscr{X} \times \mathscr{Y}) \to \mathscr{G}(\mathscr{X} \times \mathscr{Y})$ *defined by*

$$\tilde{\mathscr{T}} = \left\{ \tilde{T} \colon \tilde{T} f = \sum_{y\in\mathscr{Y}} I_y T_y f(\cdot, y), T_y \in \mathscr{T}_\mathscr{X} \forall y \in \mathscr{Y} \right\} \tag{23}$$

is called an additive independent extension *of* $\mathscr{T}_\mathscr{X}$ *from* $\mathscr{G}(\mathscr{X})$ *to* $\mathscr{G}(\mathscr{X} \times \mathscr{Y})$.

Definition 23 *Let* $\mathscr{T}_\mathscr{X}$ *be as in Definition 22. Then the extension* $\mathscr{T}^E = \tilde{\mathscr{T}} \cup \mathscr{P}(\mathscr{Y})$ *is called* epistemically irrelevant extension *of* $\mathscr{T}_\mathscr{X}$ *from* $\mathscr{G}(\mathscr{X})$ *to* $\mathscr{G}(\mathscr{X} \times \mathscr{Y})$.

Corollary 24 *Let* $\mathscr{D} \subset \mathscr{G}(\mathscr{X} \times \mathscr{Y})$ *be a set of desirable gambles that is an AIE of* $\mathscr{D} \cap \mathscr{G}(\mathscr{X}|y)$. *Then*

(i) $\mathscr{D}$ *is generated by* $\tilde{\mathscr{T}}$ *if and only if* $\mathscr{D}$ *is a marginal extension of some* $\mathscr{D}(\mathscr{Y})$ *and* $\mathscr{D}(\mathscr{X}|y)$ *for* $y \in \mathscr{Y}$;

(ii) $\mathscr{D}$ *is generated by* $\mathscr{T}^E$ *if and only if it satisfies epistemic irrelevance* $\mathscr{Y} \to \mathscr{X}$ *and is therefore a marginal extension of some* $\mathscr{D}(\mathscr{Y})$*, satisfying symmetry, and* $\mathscr{D}(\mathscr{X}|y)$ *for* $y \in \mathscr{Y}$.

Example 2 *Consider the case where* $\mathscr{X} = \mathscr{Y} = \{0,1\}$. *Let* $\mathscr{D}(\mathscr{Y}) = \text{posi}(\{(-1,2),(2,-1)\} \cup \mathbb{R}^2_+)$ *and* $\mathscr{D}(\mathscr{X}) = \mathbb{R}^2_{\geq 0}$. *Then the natural extension of* $\mathscr{D}(\mathscr{Y})$ *satisfies epistemic irrelevance* $\mathscr{Y} \to \mathscr{X}$.

4.2. Epistemic Irrelevance with Additional Variables

The more general case where epistemic irrelevance is possible is where we consider gambles that depend on three (or more) variables. Thus let $\mathscr{X}, \mathscr{Y}$ and $\mathscr{Z}$ be given and suppose that the probabilistic model on $\mathscr{Y}$ is irrelevant to $\mathscr{X}$. Thus we have local models on $\mathscr{X}$ that possibly depend on $\mathscr{Z}$. In general we thus have sets of Markov operators

$$\mathscr{T}_{(y,z)} \colon \mathscr{G}(\mathscr{X}|(y,z)) \to \mathscr{G}(\mathscr{X}|(y,z)), \tag{24}$$

where $\mathscr{T}_{(y,z)} = \mathscr{T}_z$ for every $y \in \mathscr{Y}$. We would like to extend these models to a global model on $\mathscr{G}(\mathscr{X} \times \mathscr{Y} \times \mathscr{Z})$ satisfying epistemic irrelevance of $\mathscr{X}$ from $\mathscr{Y}$.

As before, in addition to sets of operators being independent of $\mathscr{Y}$, we must also ensure symmetry, which we do using permutation operators on $\mathscr{Y}$. Thus, for any permutation σ of elements in $\mathscr{Y}$ we define

$$P^\sigma f(x,y,z) = f(x,\sigma(y),z), \tag{25}$$

for every $f \in \mathscr{G}(\mathscr{X} \times \mathscr{Y} \times \mathscr{Z})$. The set of all permutation operators will be again denoted by $\mathscr{P}(\mathscr{Y})$. First we use AIE to extend the sets of operators $\mathscr{T}_{(y,z)}$ from $\{\mathscr{G}(\mathscr{X}|(y,z))\}_{z\in\mathscr{Z}}$ to $\mathscr{G}(\mathscr{X} \times \mathscr{Z}|y)$:

$$\tilde{\mathscr{T}}_y \colon \mathscr{G}(\mathscr{X} \times \mathscr{Z}|y) \to \mathscr{G}(\mathscr{X} \times \mathscr{Z}|y)$$

where

$$\tilde{T}_y f = I_y \sum_{z\in\mathscr{Z}} I_z T_{(y,z)} f(\cdot,y,z)$$

for every $f \in \mathscr{G}(\mathscr{X} \times \mathscr{Z}|y)$, where $T_{(y,z)} \in \mathscr{T}_z$. With $\tilde{\mathscr{T}}_y$ we denote the set of all operators of the above form. Finally we use AIE again to extend $\{\tilde{\mathscr{T}}_y\}_{y\in\mathscr{Y}}$ to $\mathscr{G}(\mathscr{X} \times \mathscr{Y} \times \mathscr{Z})$ to obtain the set $\tilde{\mathscr{T}}$. Then we additionally require that the extension is generated by $\mathscr{P}(\mathscr{Y})$.

Theorem 25 *Let $\mathscr{T}_z$ for each $z \in \mathscr{Z}$ be a collection of sets of Markov operators acting on $\mathscr{G}(\mathscr{X})$. Further let $\mathscr{T}_{(y,z)} = \mathscr{T}_z$ respectively act on spaces $\mathscr{G}(\mathscr{X}|(y,z))$. For each $y \in \mathscr{Y}$ let $\mathscr{D}_y \subseteq \mathscr{G}(\mathscr{X} \times \mathscr{Z}|y)$ be a set of desirable gambles that is an AIE of $\{\mathscr{T}_{(y,z)}\}_{z\in\mathscr{Z}}$. Further, let $\mathscr{D}$ be an AIE of $\{\mathscr{D}_y\}_{y\in\mathscr{Y}}$ and generated by $\tilde{\mathscr{T}}$ and $\mathscr{P}(\mathscr{Y})$. Then $\mathscr{D}$ satisfies epistemic irrelevance $\mathscr{Y} \to \mathscr{X}$.*

Proof Take some $f \in \mathscr{G}(\mathscr{X})$. Then we can write $f = \sum_{y\in\mathscr{Y}} I_y f$. The set $\mathscr{D}$ satisfies epistemic irrelevance if the following equivalence holds: $f \in \mathscr{D}$ if and only if, for all $y \in \mathscr{Y}$, $I_y f \in \mathscr{D}$. Note that the 'if' part follows immediately by the convexity of $\mathscr{D}$.

Now we prove the 'only if' part. Since $\mathscr{D}$ is an AIE of $\{\mathscr{D}_y\}_{y\in\mathscr{Y}}$, $f \in \mathscr{D}$ must imply that $I_y f$ is desirable for at least one $y \in \mathscr{Y}$. Now, since $\mathscr{D}$ is also generated by the set of permutation operators, we have that $P^\sigma[I_y f] = I_{\sigma(y)} f \in \mathscr{D}$ as well. Because this holds for every permutation σ, then clearly all $I_y f$ must be desirable. ∎

Corollary 26 *Let $\mathscr{D} = \mathscr{D}(\mathscr{X} \times \mathscr{Y} \times \mathscr{Z})$ satisfy the conditions of Theorem 25. Then there exist sets of desirable gambles $\mathscr{D}(\mathscr{Y}), \mathscr{D}(\mathscr{Z})$ and $\mathscr{D}_z(\mathscr{X})$ for every $z \in \mathscr{Z}$ so that $\mathscr{D}(\mathscr{Y})$ is generated by $\mathscr{P}(\mathscr{Y})$ and $\mathscr{D}_z(\mathscr{X})$ by respective $\mathscr{T}_z$, so that*

$$\mathscr{D} = \Big\{ f_{\mathscr{Y}} + \sum_{y\in\mathscr{Y}} I_y (f_y + \sum_{z\in\mathscr{Z}} I_z f_{(y,z)}) \colon \\ f_{\mathscr{Y}} \in \mathscr{D}(\mathscr{Y}), f_y \in \mathscr{D}(\mathscr{Z}), f_{(y,z)} \in \mathscr{D}_z(\mathscr{X}) \Big\}.$$

Proof The existence of sets $\mathscr{D}(\mathscr{Y})$ and $\mathscr{D}_y(\mathscr{Z})$, that could in principle depend on $y \in \mathscr{Y}$, and $\mathscr{D}_{(y,z)}(\mathscr{X})$, that could also depend on y, follows from the above shown properties of additive independent extensions. The fact that the sets of desirable gambles on $\mathscr{Z}$ and $\mathscr{X}$ do not depend on y follows from the requirement that $\mathscr{D}$ is also generated by $\mathscr{P}(\mathscr{Y})$ which ensures symmetry with respect to $\mathscr{Y}$. ∎

5. Conclusions and Further Work

Sets of Markov operators can in some cases present a more general alternative to imprecise probability models. This paper provides extensions of such sets from collections of subspaces to larger spaces. These extensions are consistent with a new type of extensions of imprecise probabilistic models, called additive independent extension. Moreover, we have applied the new type of extensions to construct epistemically irrelevant extensions of marginal models to product spaces.

The types of independence used in this paper are very common, among others, in the theory of stochastic processes [6, 8], especially the model described in Section 4.2. Specifically, epistemic irrelevance of a variable Y to X may be used to model irrelevance of the process history to its future evolution, while Z denotes the current state, which does influence future behaviour. The question of constructing global models from local ones is of great importance in the theory of stochastic processes. It is therefore one of the plans for future work to describe the models of imprecise stochastic processes using the approaches presented in this paper.

Furthermore, the notion of epistemic irrelevance used here is described as *epistemic value-irrelevance* in [2], where also a stronger notion of *epistemic subset-irrelevance* is used. Another future plan is therefore to explore whether this stronger property can be described in a similar way with extensions of sets of Markov operators.

Acknowledgments

1. The author acknowledges the financial support from the Slovenian Research Agency (research core funding No. P5-0168).

2. The author wishes to thank three anonymous referees for their comments, which helped to clarify and improve the paper.

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