Robust Uncertainty Quantification for Measurement Problems with Limited Information

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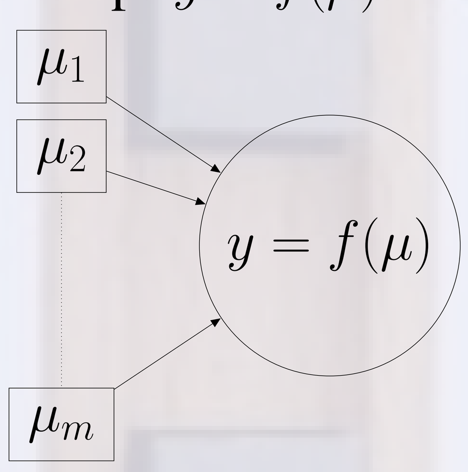
Introduction

- We investigate the use of imprecise probability for metrology.
- Uncertainty quantification of end-gauge calibration process.
- Limited data.
- Lack of knowledge regarding the source of uncertainty.
- Lack of knowledge of dependencies between measurements.

Model and Objective [2]

Input-output model:

- Uncertain inputs: $\mu := (\mu_1, \cdots, \mu_m)$
- Quantity of interest (output): y
- Known functional relationship: $y = f(\mu)$:



Challenges: The measurements are noisy and limited.
Objective: Obtain a robust estimate of $f(\mu)$ based on an estimate of μ

Delta Method [3]

Let $\hat{X} := (\hat{X}_1, \cdots \hat{X}_m)$ be an estimator of μ.
Assume $\hat{X} \sim N(\mu, \Sigma)$ (approximately).
Method:

- Taylor expand f, such that

$$f(\hat{X}) \approx f(\mu) + \nabla f(\mu)^T(\hat{X} - \mu)$$

- Compute the covariance matrix of $f(\hat{X})$,

$$\text{Cov}(f(\hat{X})) := \nabla f(\mu)^T \Sigma \nabla f(\mu)$$

- Obtain a confidence interval around $f(\mu)$ using the relation:

$$f(\hat{X}) \sim N(f(\mu), \nabla f(\mu)^T \Sigma \nabla f(\mu))$$

Limitations:

- f must be differentiable with respect to the input variables.
- f has to be approximately linear around μ for the distributional range of $\hat{X}$.
- $f(\hat{X})$ may not be Gaussian, when f is highly non-linear.
- We may have to use sample standard deviation instead of Σ and we may have to use $\nabla f(\hat{X})$ instead of $\nabla f(\mu)$.

P-box [1]

P-box: A p-box is specified by two cumulative distribution functions $\underline{F}$ and $\overline{F}$, and contains the set of all cumulative distribution functions bounded by $\underline{F}$ and $\overline{F}$:

$$\{F \in \mathcal{F} \colon \underline{F}(x) \le F(x) \le \overline{F}(x), \forall x \in \mathbb{R}\}.$$

- Easy propagation through non-linear operators.
- Relaxes/removes distributional assumption.
- Allows to relax assumptions about dependence.

Background image : `https://www.npl.co.uk/special-pages/guides/gpg149_gauge`

Example: End gauge calibration [2]

Problem: Estimate length (ℓ_M) of an end gauge (M) by comparing it with length (ℓ_S) of a known standard (S) using the relation:

$$\ell_M = \frac{\ell_S(1 + \alpha_S \theta_S) + d}{1 + \alpha_M \theta_M}$$

where, α_M and θ_M (α_S and θ_S) are thermal expansion coefficient and temperature deviation of M (S) and d is the difference between ℓ_M and ℓ_S. Linearization gives:

$$\ell_M = \ell_S + d - \ell_S(\delta\alpha \cdot \theta_M - \alpha_S \cdot \delta\theta)$$

where, $\delta\alpha = \alpha_M - \alpha_S$ and $\delta\theta = \theta_M - \theta_S$.

quantity	sample mean	sample standard deviation	units
d^*	0	0.97e-06	cm
ℓ_S^*	50	2.5e-06	cm
α_M	11.5e-06	1.33e-06	Celsius^{-1}
α_S	11.5e-06	1.2e-06	Celsius^{-1}
$\delta\alpha$	0	0.58e-06	Celsius^{-1}
θ_M	20	0.41	Celsius
θ_S	20	0.411	Celsius
$\delta\theta$	0	0.029	Celsius

Estimates: We compare inferences from delta method and p-box method, under various dependence assumptions.

Method	mean	variance
Δ - method	50	1.156e-11
Independence (all)		
Gaussian	50.0000000410598	[2.8239e-06, 4.4617e-06]
Distribution free	[49.99191, 50.00807]	[0, 0.00034]
Frechet (ℓ_S, d)		
Gaussian	[49.99995, 50.00005]	[2.8239e-06, 4.4617e-06]
Distribution free	[49.99188, 50.00809]	[0, 0.00034]
Frechet (α's, θ's)		
Gaussian	[49.9988, 50.0012]	[7.6372e-07, 8.9590e-06]
Distribution free	[49.97665, 50.02333]	[0, 0.0014]
Frechet (all)		
Gaussian	[49.99879, 50.00121]	[7.6372e-07, 8.9590e-06]
Distribution free	[49.97664, 50.02334]	[0, 0.0014]

Discussion

- We investigate the use of p-boxes to propagate uncertainty in measurement problems.
- We illustrate our approach using an end gauge calibration problem.
- We compare our result with the classical delta method.
- Using p-boxes, we can relax distributional assumptions.

References

[1] Scott Ferson et al. 'Constructing probability boxes and Dempster-Shafer structures'. In: *Sandia journal manuscript; Not yet accepted for publication* (May 2015). ISSN: 9999-0014.

[2] *Evaluation of measurement data – Guide to the expression of uncertainty in measurement*. Accessed: 2019-05-24. 2008.

[3] Aad W. van der Vaart and Jon A. Wellner. *Weak Convergence and Empirical Processes: With Applications to Statistics*. Springer New York, 1996. DOI: `10.1007/978-1-4757-2545-2`.

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