Imprecise multivariate analysis under limited data

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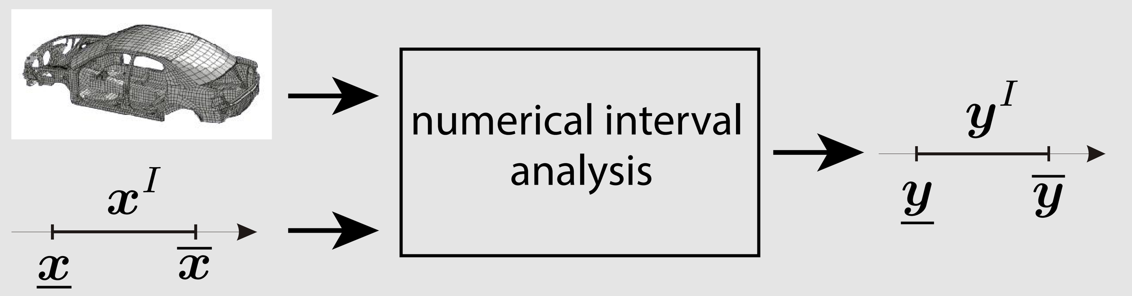
Introduction

This poster highlights recent developments in the field of modelling multivariate and spatial uncertainty under scarce data in the context of interval and imprecise probabilistic analysis. Specific problems with respect to independence in intervals are tackled for both frameworks.

Hereto, new methods for the modelling of dependence between multiple intervals and interval fields are presented, as well as a framework for the definition and modelling of random fields with imprecision in both the central moments, as in the covariance structure. Case studies are included to illustrate the developed methods.

Interval Finite Element analysis



Interval-valued model parameters

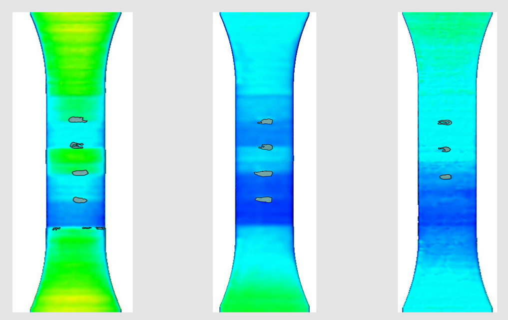
Bounds on model responses

The (in)dependence problem

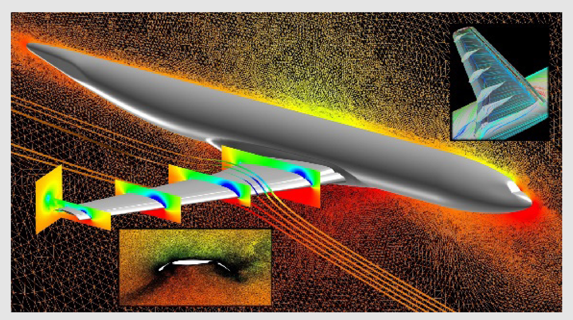
Interval vector x^I describes hypercube in $\mathbb{R}^{d_x}$: $x^I = x_1^I \times x_2^I \times \cdots \times x_{d_x}^I$

$\rightsquigarrow$ By definition independent!

How to model multivariate uncertainty in this case? E.g.,:



Non-random spatial uncertainty



Complex coupled phenomena

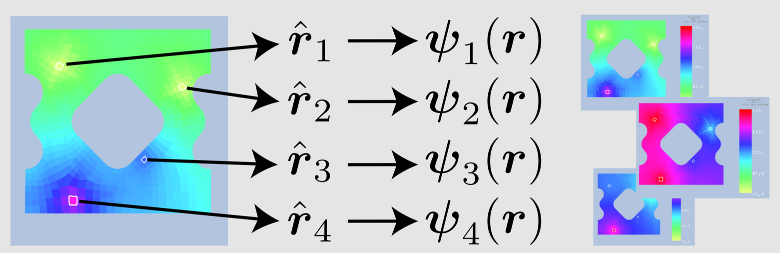
Interval fields

$$\boldsymbol{x}^I(\boldsymbol{r}) = \sum_{i=1}^{n_b} \alpha_i^I \psi_i(\boldsymbol{r})$$

→ Basis functions, computed via:
- Maximum gradient method
- Inverse distance weighting:

$$\psi_i(\boldsymbol{r}) = \frac{[d(\hat{\boldsymbol{r}}_i, \boldsymbol{r})]^{-p}}{\sum_{j=1}^{n_b}[d(\hat{\boldsymbol{r}}_j, \boldsymbol{r})]^{-p}}$$

↓ Independent interval scalars



Multivariate dependent interval analysis

Dependence between two intervals is modelled by an admissible set $\mathcal{D}$:

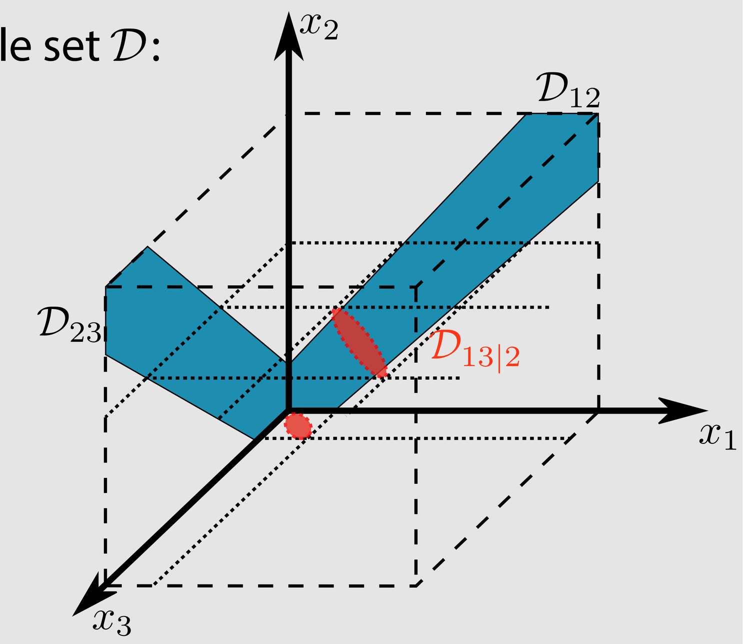
$$\mathcal{D} \subseteq \boldsymbol{x}^I = x_1^I \times x_2^I \times \cdots \times x_{d_i}^I$$

How can we model $\mathcal{D}$ in general d_i dimensional space?

$\rightsquigarrow$ Probabilistic context: copula pair constructions (CPC)

Idea: translate CPC towards set-theoretical framework for the definition of dependence between multiple intervals:

$$\mathcal{D}_{1,\ldots,k}(x_1^I, x_2^I, \ldots, x_k^I) = \bigotimes_{k=1}^{d_i} x_k^I \bigcap_{j=1}^{d_i-1} \bigcap_{i=1}^{d_i-j} \mathcal{D}_{i,i+j|i+1,\ldots,i+j+1}$$



Ref: M. Faes and D. Moens, Comput. Methods Appl. Mech. Eng., vol. 347, pp. 85–102

Cross-dependent interval field analysis

How to model cross-interdependence between two interval fields ${}^{(1)}\boldsymbol{x}^I(\mathbf{r})$ and ${}^{(2)}\boldsymbol{x}^I(\mathbf{r})$?

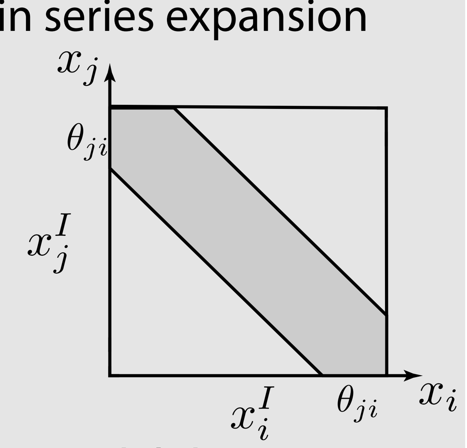
$${}^{(1)}\boldsymbol{x}^I(\mathbf{r}) = \sum_{i=1}^{n_{b1}} \alpha_i^I \, {}^{(1)}\boldsymbol{\psi}_i(\mathbf{r}) \qquad {}^{(2)}\boldsymbol{x}^I(\mathbf{r}) = \sum_{i=1}^{n_{b2}} \beta_i^I \, {}^{(2)}\boldsymbol{\psi}_i(\mathbf{r})$$

Idea: apply multivariate dependent interval analysis to couple interval scalars in series expansion via a diagonal band admissible set, which is in case of negative dependence:

$$\mathcal{D}_{ij} = 1 - \mathcal{H}(|x_i + x_j - 1| - \theta_{ji}) \qquad \forall x_i, x_j \in \boldsymbol{x}^I$$

yielding:

$$\mathcal{D}(\alpha_l^I, \beta_1^I, \beta_2^I, \ldots, \beta_{n_{b2}}^I) = \bigotimes_{k=1}^{n_{b2}+1} x_{k,l}^I \bigcap_{i=2}^{n_{b2}+1} 1 - \mathcal{H}(|x_1 + x_i - 1| - \theta_{1i})$$



This limits the joint admissible set of both interval fields for each location in the model domain, as illustrated in the example.

Imprecise random field analysis

Imprecise random field via Karhunen-Loeve given intervals on mean,correlation length and variance:

$$[x](\mathbf{r}, \theta) = \mu_x^I(\mathbf{r}) + \sigma_x^I \sum_{i=1}^{M} \sqrt{\lambda_i^I} \boldsymbol{\psi}_i^I(\mathbf{r}) \xi_i(\theta)$$

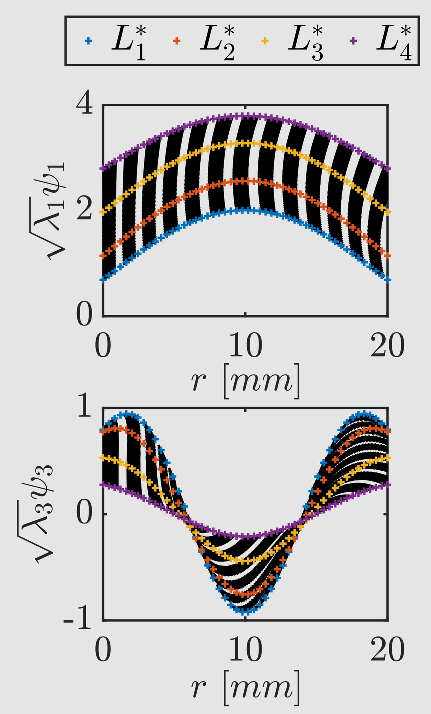
Each stochastic realisation is as such an interval field and each interval realisation is a random field.

Problem: not all realisations in $\sqrt{\lambda_i^I} \psi_i^I(\boldsymbol{r})$ form a complete Karhunen-Loeve basis due to to the (in)dependence problem of intervals

Idea: apply an iterative solution that ensures a complete basis:

$$\underline{L}_i^* = \underset{\mathcal{G}(L)}{\arg\min} ||\sqrt{\lambda_i}\psi_i(\mathbf{r})||_2, \qquad \text{s.t. } L \in L^I, i = 1, \cdots, M$$

$$\overline{L}_i^* = \underset{\mathcal{G}(L)}{\arg\max} ||\sqrt{\lambda_i}\psi_i(\mathbf{r})||_2, \qquad \text{s.t. } L \in L^I, i = 1, \cdots, M$$



This yields set of correlation lengths $\mathcal{L}^*$ that provide the bounds on the responses of monotonic models, as illustrated in the example of the car dynamics.

Presenter contact information

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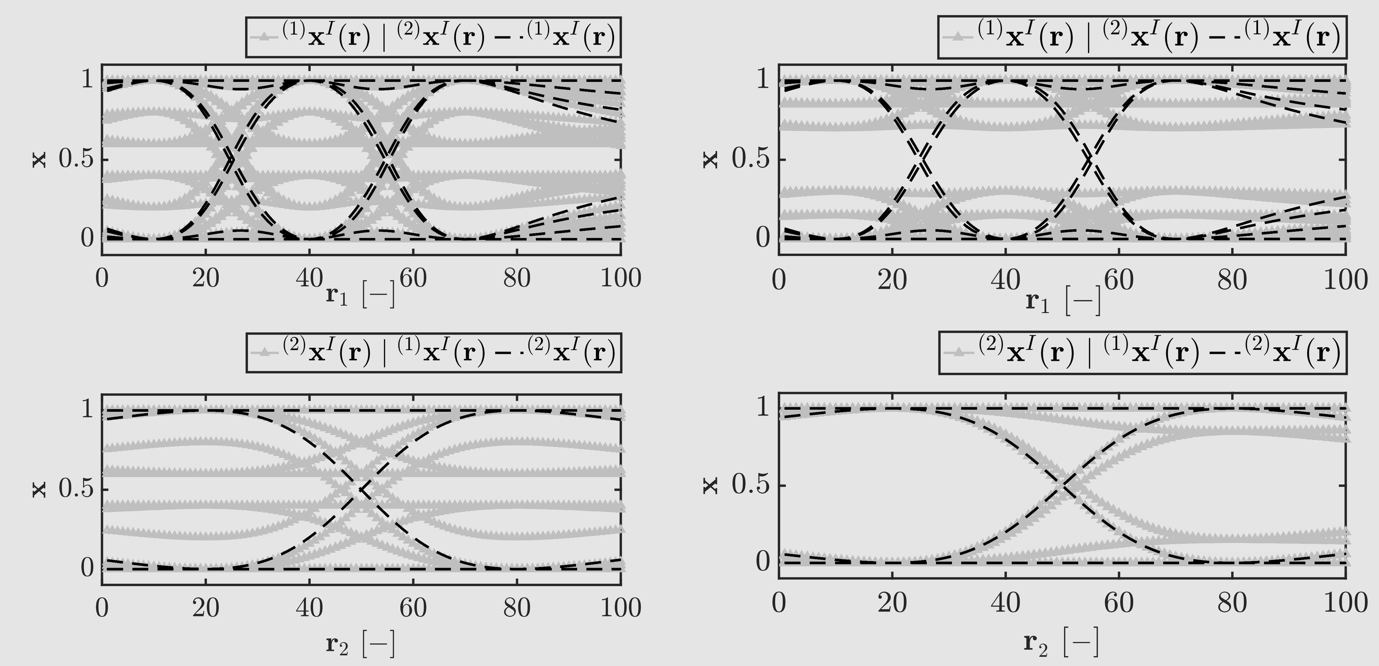
Presenter bio

Illustration of cross-dependent interval fields

One dimensional domain Ω that is discretized into 10 elements Ω_e. On this domain, two interval fields with respectively 3 and 2 control points are defined and their cross-dependence is modelled using the presented approach using the admissible set decomposition coupling.

Both homogeneous (left), i.e., all θ_{ji} are equal, and heterogeneous (right) cross dependence are illustrated.

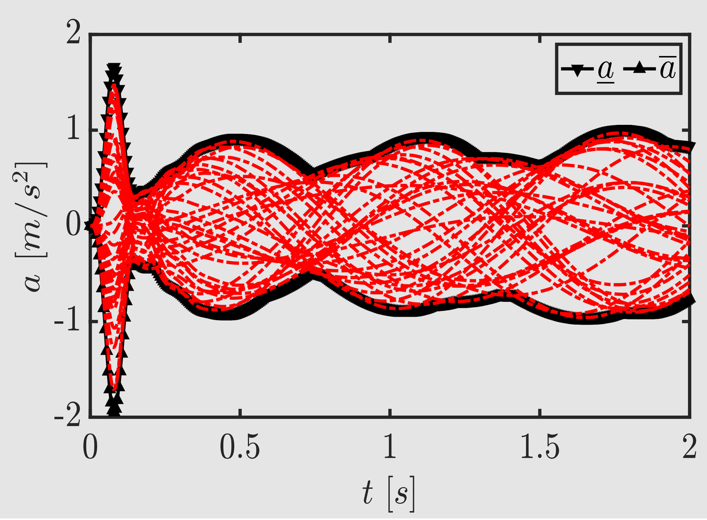


Application example: Car Dynamics

Car dynamics are modelled as a damped state-space model. The base excitation is modelled as zero-mean imprecise Gaussian random field with interval-valued correlation length and standard deviation:

$\sigma^I = [0.0015; 0.003]$ m and $L^I = [2; 15]$ m

The output quantity of interest is the acceleration of the sprung mass.



$$\underline{a}(t_i) = \min a(t_i \mid \mathcal{L} \cup \mathcal{T}, \theta)$$
$$\overline{a}(t_i) = \max a(t_i \mid \mathcal{L} \cup \mathcal{T}, \theta)$$
$$\forall t_i \in [0, 2] \ s, \forall \theta_i, i = 1, \ldots, N$$