The Ergodic conundrum

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1. Motto

"No probability without ergodicity"

Nassim Taleb, *Skin in the game* (2018)

2. Main questions

◇ $f : M \to M$ is a map defined on a compact topological (metric) space M and $A \subset M$.

Can we give mathematical *sense* to the questions:

Given a point $x \in M$ what is the probability of that point visiting the set A?
&
Without knowing the initial state what is the probability of the set A being visited?

It is not the aim of this work to give a definitive answer to these questions.
Rather we will try to make them more
IMPRECISE...!

7. Probability... imprecisely

We interpret the "probability" of visiting a given set A as being

◇ initial state of the system, x say, is known:

$$\Pr_x(A) := [\ell_x(A), u_x(A)]$$

◇ initial state of the system is unknown:

$$\Pr(A) := [\inf_{x\in M}\{\ell_x(A)\}, \sup_{x\in M}\{u_x(A)\}]$$

▽ If the phase space contains two (open) forward-invariant sets A and B such that $A \cap B = \emptyset$ then for both sets we have that

$$\Pr(A) = \Pr(B) = [0, 1]$$

which is *empty* information.

SO, there must be some form of indecomposability or restriction of phase space associated with this definition of probability.

8. Special case

◇ λ is a *a priori* defined probability on a σ-algebra of the space M (e.g., a normalised Liouville measure)

Alternatively, we can define

$$\Pr(A) := \left[\int_M \ell_x(A)\,d\lambda, \int_M u_x(A)\,d\lambda\right]$$

△ If λ is f-invariant itself then

$$\int_M \ell_x(A)\,d\lambda = \int_M u_x(A)\,d\lambda = \lambda(A)$$

We can also define the "distribution functions"

$$G\ell_A(s) = \lambda\{x : \ell_x(A) > s\}$$
$$Gu_A(s) = \lambda\{x : u_x(A) > s\}$$

This allows for a representation of the "probability" of visiting a set A as a set of functions, i.e.

$$\{g : [0,1] \to [0,1] : G\ell_A(s) \le g(s) \le Gu_A(s)\}$$

3. An application of Birkhoff's theorem

◇ probability measure μ defined on a σ-algebra $\mathcal{B} \subset \mathcal{P}(M)$ is **invariant** under transformation f if $\mu(A) = \mu(f^{-1}(A))$ for every $A \in \mathcal{B}$.

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbf{I}_A\left(f^k(x)\right) \underset{n\to\infty}{\longrightarrow} \text{freq}_A(x)$$

for μ-almost every point x and moreover

$$\mu(A) = \int \text{freq}_A(x)\; d\mu$$

▽ the set of points which do not have converging frequencies of visits to a given set is neglected from the point of view of the *given* measure

▽ The probability of visiting a set cannot be interpreted as an *average* of the convergent frequencies because this average is calculated with respect to μ

SO, the Ergodic theorem of Birkhoff should not be used as a means of interpreting the probability of a physical system visiting a set.

6. Modularity of ℓ_x and u_x

By a well known representation result (see [1]) there exist two sets of finitely-additive measures F^* and F_* such that

$$L_x(\varphi) = \inf_{\mu\in F_*}\left\{\int \varphi\, d\mu\right\}$$

$$U_x(\varphi) = \sup_{\nu\in F^*}\left\{\int \varphi\, d\nu\right\}$$

In particular:

$$\ell_x(A) = \inf_{\mu\in F_*}\{\mu(A)\}$$

$$u_x(A) = \sup_{\nu\in F^*}\{\nu(A)\}$$

Using a characterisation result of sub-(resp. super)-modular set functions (see [2]) we conclude that

ℓ_x is *supermodular*:

$$\ell_x(A \cup B) + \ell_x(A \cap B) \ge \ell_x(A) + \ell_x(B)$$

u_x is *submodular*:

$$u_x(A \cup B) + u_x(A \cap B) \le u_x(A) + u_x(B)$$

Note: For any set function ν represented as a lower envelope of a family of capacities, F_*, we have that F_* can be chosen as core(ν) where

$$\text{core}(\nu) = \{P : P \ge \nu\}$$

9. In a very special case...

...we can prove a version of the Khintchine recurrence theorem:

Let ν be an f-invariant set function such that $\nu(A) = \inf_{\mu\in\text{core}(\nu)}\{\mu(A)\}$ and core(ν) is contained in the set of invariant probabilities of f. Then for every $0 \le \alpha < 1$ the set

$$\{n : \overline{\nu}(A \cap f^{-n}(A)) > \alpha\nu(A)^2\}$$

is syndetic.

4. Not all sets are welcome!

◇ point x which is not eventually periodic
◇ sequence $\sigma = \sigma_1\sigma_2\cdots\sigma_k\cdots$; $\sigma_n \in \{0,1\}$

$$A_\sigma = \left\{f^k(x) : \sigma_k = 1; k \in \mathbb{N}\right\}$$
$$\Downarrow$$
$$\sharp\left\{k : f^k(x) \in A_\sigma\right\} = \sharp\{k : \sigma_k = 1\}$$

∴ Any sequence of visits can be produced for any such x.

SO, consider open and connected sets?
e.g. open balls in the case of a metric space

5. Upper and Lower operators

◇ $x \in M$ and $\varphi : M \to \mathbb{R}$ bounded function

lower time-average
$$L_x(\varphi) := \liminf_{n\to\infty} \tfrac{1}{n}\textstyle\sum_{k=1}^{n} \varphi\left(f^k(x)\right)$$

upper time-average
$$U_x(\varphi) := \limsup_{n\to\infty} \tfrac{1}{n}\textstyle\sum_{k=1}^{n} \varphi\left(f^k(x)\right)$$

L_x and U_x are both:

- *monotone*: $\varphi_1 \le \varphi_2$ implies $\Gamma(\varphi_1) \le \Gamma(\varphi_2)$
- *positive homogeneous*:
  $\Gamma(\lambda\varphi) = \lambda\Gamma(\varphi), \lambda > 0$
- *translation invariant*:
  $\Gamma(\varphi + c) = \Gamma(\varphi) + c, c \in \mathbb{R}$
- *f-invariant*: $\Gamma(\varphi \circ f) = \Gamma(\varphi)$
- L_x is *superadditive*:
  $L_x(\varphi_1 + \varphi_2) \ge L_x(\varphi_1) + L_x(\varphi_2)$
- U_x is *subadditive*:
  $U_x(\varphi_1 + \varphi_2) \le U_x(\varphi_1) + U_x(\varphi_2)$

upper and **lower** frequencies
$\ell_x(A) := L_x(\mathbf{I}_A)$ and $u_x(A) := U_x(\mathbf{I}_A)$

ℓ_x and u_x are set functions satisfying:

- $\ell_x(\emptyset) = u_x(\emptyset) = 0$ and $\ell_x(M) = u_x(M) = 1$
- u_x is the dual set function of ℓ_x that is
  $\ell_x(A) + u_x(A^c) = 1,$
  (where A^c is the complementary set of A)
- if $A \subset B$ then
  $\ell_x(A) \le \ell_x(B)$ and $u_x(A) \le u_x(B)$
- f-invariance:
  $\ell_x(A) = \ell_x\left(f^{-1}(A)\right)$ and $u_x(A) = u_x\left(f^{-1}(A)\right)$

10. Poincaré recurrence generalised

Given a f-invariant superadditive capacity ν (with $\nu(M) < \infty$) defined on $S \subset 2^M$ on which f is measurable then for every $A \in S$ with $\nu(A) > 0$ and $n \in \mathbb{N}$ the set of points that do not return to A after n iterates has measure zero.

References

[1] Föllmer, Hans and Schied, Alexander (2004). *Stochastic Finance: An Introduction in Discrete Time*, Walter de Gruyter.

[2] Denneberg, Dieter (1994). *Non-Additive Measure and Integral*, Springer.

Funding

Supported by CMUP funded by Fundação para a Ciência e a Tecnologia (FCT) ref. UID/MAT/00144/2019, Portugal.