Improving and benchmarking of algorithms for decision making with lower previsions

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Decision making with lower previsions

Let Ω denote the set of states of nature, and let $\mathcal{L}$ denote the set of all gambles (i.e. bounded real-valued functions) on Ω. A *lower prevision* $\underline{P}$ is a real-valued function on $\operatorname{dom}\underline{P} \subset \mathcal{L}$, where $\operatorname{dom}\underline{P}$ is the domain of $\underline{P}$. The conjugate upper prevision $\overline{P}$ on $\operatorname{dom}\overline{P} := \{-f : f \in \operatorname{dom}\underline{P}\}$ is defined by $\overline{P}(f) := -\underline{P}(-f)$. We can extend $\underline{P}$ to the set of all gambles $\mathcal{L}$ via its *natural extension* $\underline{E}$ which is finite, and therefore, is a lower prevision, if an only if $\underline{P}$ *avoids sure loss* [3]. In the case that both Ω and $\operatorname{dom}\underline{P}$ are finite, for any f, $\underline{E}(f)$ can be calculated by solving a linear program [4].

Strict partial orderings: for any two gambles f and g, we say that
- $f \succ g$ whenever $\underline{E}(f-g) > 0$.
- $f \sqsupset g$ whenever $\underline{E}(f) > \overline{E}(g)$.

Set of maximal gambles: let $>$ be a strict partial order on $\mathcal{L}$, and let $\mathcal{K}$ be a finite subset of $\mathcal{L}$. The *set of maximal gambles in* $\mathcal{K}$ *with respect to* $>$ is then defined by:

$$\operatorname{opt}_{>}(\mathcal{K}) := \{f \in \mathcal{K} : (\forall g \in \mathcal{K})(g \not> f)\}. \tag{1}$$

- $\operatorname{opt}_{\succ}(\mathcal{K})$ the set of *maximal* gambles in $\mathcal{K}$.
- $\operatorname{opt}_{\sqsupset}(\mathcal{K})$ the set of *interval dominant* gambles in $\mathcal{K}$.

Applying interval dominance: as every maximal gamble in $\mathcal{K}$ is interval dominant, we can eliminate non-maximal gambles in $\mathcal{K}$ by applying interval dominance first [2].

E-admissibility: f is *E-admissible* in $\mathcal{K}$ if there is a p in the credal set of $\underline{P}$ such that

$$\forall g \in \mathcal{K}: E_p(f) \geq E_p(g). \tag{2}$$

Note that any E-admissible gamble f in $\mathcal{K}$ is also maximal [5, §3.9.4].

An overview of the structure of this study

Fast evaluation of natural extensions inside algorithms

Jansen et al. [1] (Alg 3) — Troffaes and Hable [4] (Alg 1) → Our algorithm (Alg 2) ← Sorting gambles

Improving Jansen et al. [1] (Alg 4) → Compare their performance

Apply interval dominance

Benchmarking framework ← Generating sets of gambles with *pre-specified ratios of maximal and interval dominant gambles*

Fast evaluation of natural extensions inside algorithms

- **Checking maximality**: to check maximality of f, one has to evaluate the sign of the lower prevision of several gambles. Specifically, if there exists g such that $\underline{E}(g-f) > 0$, then f is not maximal. Note that $\underline{E}(g-f)$ can be calculated through solving a linear program.
- **Fast evaluation of natural extensions inside algorithms**:
  - **Common feasible starting points**.
  - **Early stopping criteria** to determine the sign of lower previsions.

Stop if feasible value is positive. — Stop if feasible value is negative.

maximising — Optimal value ★ — minimising

Benchmarking

- **Benchmarking these four algorithms**
- **Generating sets of gambles**: that have precisely given number of maximal and interval dominant gambles.

Generating m **maximal** gambles → Generating $n-m$ **non-maximal but inerval dominant** gambles → Generating $k-n$ **not inerval dominant** gambles

- **A way to generate different types of gambles**

Shift a gamble h by α, where α is
- $\max_{f\in\operatorname{opt}_{\succ}(\mathcal{K})} \underline{E}(h-f) \leq \alpha \leq \min_{f\in\operatorname{opt}_{\succ}(\mathcal{K})} \overline{E}(h-f)$ → $h-\alpha$ is maximal
- $\min_{f\in\operatorname{opt}_{\succ}(\mathcal{K})} \overline{E}(h-f) < \alpha \leq \overline{E}(h) - \max_{f\in\mathcal{K}} \underline{E}(f)$ → not maximal but interval dominant
- $\alpha > \overline{E}(h) - \max_{f\in\mathcal{K}} \underline{E}(f)$ → not interval dominant

Improving algorithms for finding maximal gambles

Improving Troffaes and Hable [4, p. 336]'s algorithm

Troffaes and Hable [4]'s algorithm → Comparing with known maximal gambles before comparing with remaining gambles ← Fast evaluation of natural extensions inside algorithms

Sorting $f_1, \dots, f_k$ by expectation → f_k is immediately maximal and evaluating only $\underline{E}(f_j - f_i)$ when $j > i$

Our algorithm

$\exists g \in \mathcal{K}: \underline{E}(g-f) > 0$ — No: f is maximal; Yes: f isn't maximal

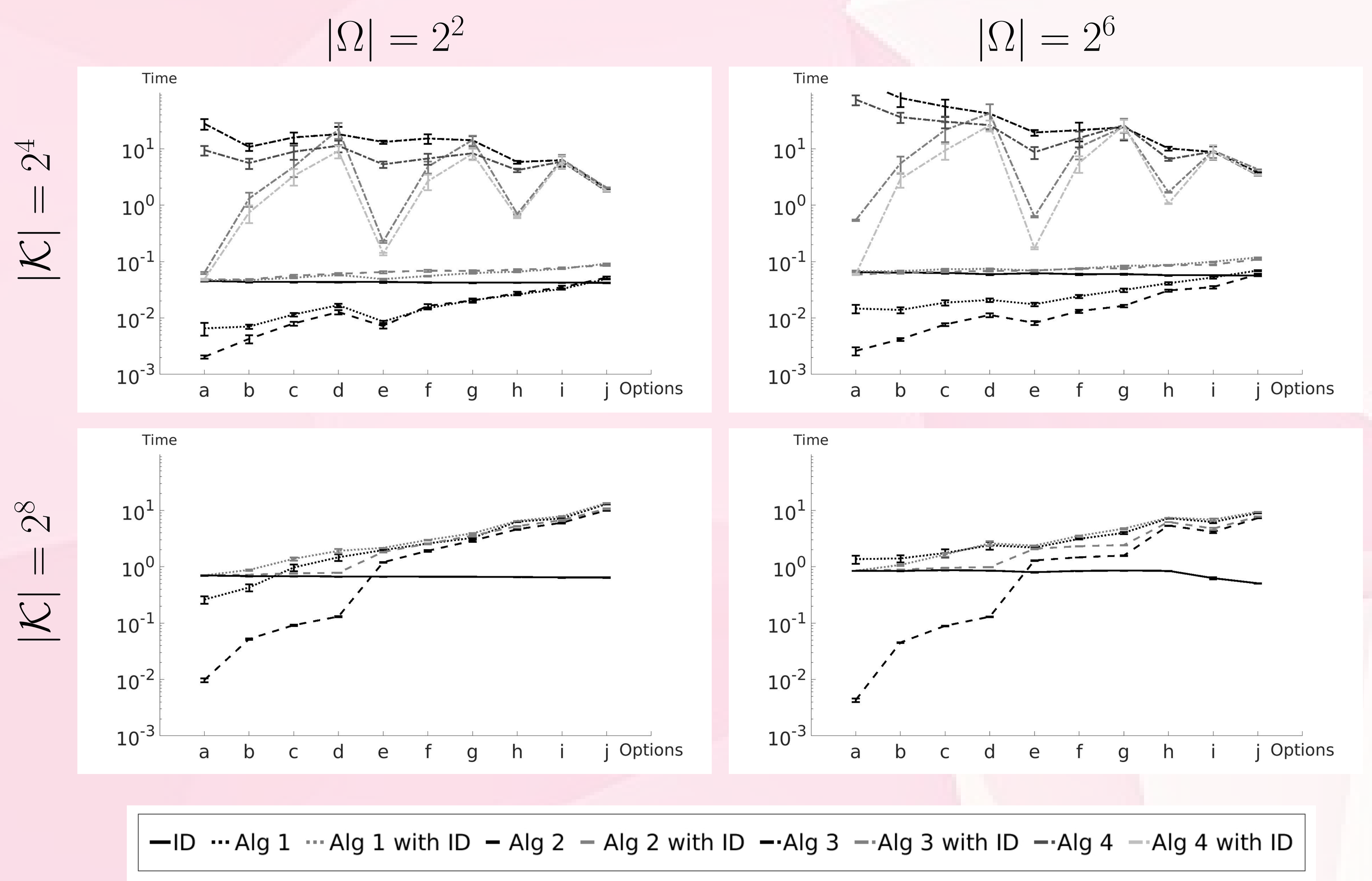
Improving Jansen et al. [1]'s algorithm

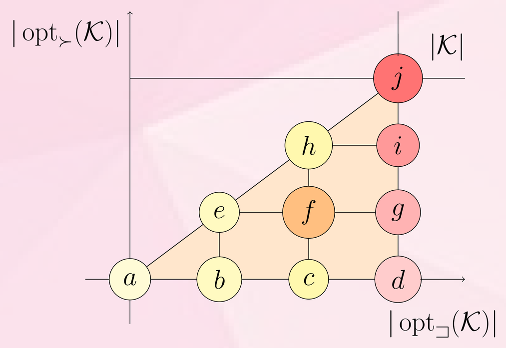
Jansen et al. [1]'s algorithm → Solving a single linear program per gamble → Feasible solutions exist — No: f isn't maximal; Yes: f is maximal

Improved Jansen et al. [1]'s algorithm → Reduce the number of constraints

Applying Troffaes and Hable [4]: if f is not maximal, then we exclude it from all further iterations

Results





Options		a	b	c	d	e	f	g	h	i	j
$\lvert\mathcal{K}\rvert = 2^4$	$\lvert\operatorname{opt}_{\succ}(\mathcal{K})\rvert$	1	1	1	1	5	5	5	11	11	16
	$\lvert\operatorname{opt}_{\sqsupset}(\mathcal{K})\rvert$	1	5	11	16	5	11	16	11	16	16
$\lvert\mathcal{K}\rvert = 2^8$	$\lvert\operatorname{opt}_{\succ}(\mathcal{K})\rvert$	1	1	1	1	85	85	85	170	170	256
	$\lvert\operatorname{opt}_{\sqsupset}(\mathcal{K})\rvert$	1	85	170	256	85	170	256	170	256	256

Remarks

1. Our benchmarking approach does have severe computational limitations, due to the need to evaluate large numbers of natural extensions. However, this work could inspire the development of further benchmarking frameworks for testing algorithms for decision making.
2. Applying interval dominance to eliminate non-maximal gambles can make the problem smaller, and this benefits Jansen et al. [1], but not the other two algorithms.
3. We find that our algorithm, without using interval dominance, outperforms all other algorithms in all scenarios in our benchmarking.

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