# On the Decomposition of Belief Functions into Simple Support Functions

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## Summary

Shafer [5] proposed to interpret belief functions as stemming from independent simple support functions (SSF), each representing a partially reliable and elementary testimony (see "Simple support function" for this representation).

Smets [6] followed in his footsteps and proposed an alternative decomposition of belief functions into independent SSF, arguing that Shafer's was not entirely satisfactory. Smets's proposal is formally elegant and has enjoyed some success. Nonetheless, it raises its own issues (see "Smets's decomposition").

In [4], both Shafer's fundamental idea of decomposing belief functions into SSF and Smets's proposal are revisited leading to a new decomposition of belief functions into SSF and a completely different perspective on Smets's proposal. In this poster, the essential aspects of these latter two contributions are presented (see, respectively, "New decomposition" and "New perspective on the weight function").

## Simple Support Function

Let $x$ be a parameter defined on a frame of discernment $\mathcal{X} = \{x_1, \ldots, x_n\}$. A mapping $m : 2^{\mathcal{X}} \to [0,1]$ such that $\sum_{A \subseteq \mathcal{X}} m(A) = 1$ is called a mass function. It is in one-to-one correspondence with the so-called belief function $bel : 2^{\mathcal{X}} \to [0,1]$ defined by $bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B)$ for all $A \subseteq \mathcal{X}$.

A mass function $m$ such that $m(\mathcal{X}) = w$ and $m(A) = 1 - w$ for some $A \subset \mathcal{X}$ and some $w \in [0,1]$, is called a simple support function (more rigorously, a simple mass function and its associated belief function is called a SSF). It is denoted by $A^w$.

$A^w$ represents a partially reliable and elementary testimony about the actual value of $x$, with $x \in A$ the testimony and $1 - w \in [0,1]$ its reliability.

## Smets's Decomposition

Smets's decomposition relies on a generalization of the SSF $A^w$ where $w \in (0, +\infty)$:

- If $w \leq 1$, $A^w$ represents a testimony "believe $x \in A$" with reliability $1 - w$;
- If $w > 1$, $A^w$ represents a testimony "do not believe $x \in A$" with reliability $1/w$ (debt of belief).

Smets shows that for any mass function $m$ such that $m(\mathcal{X}) > 0$, we have:

$$m = \bigcirc\!\!\!\!\cap_{A \subset \mathcal{X}} A^{w(A)}, w(A) \in (0, +\infty), A \subset \mathcal{X},$$

with $\bigcirc\!\!\!\!\cap$ the unnormalized Dempster's rule.

Issues:

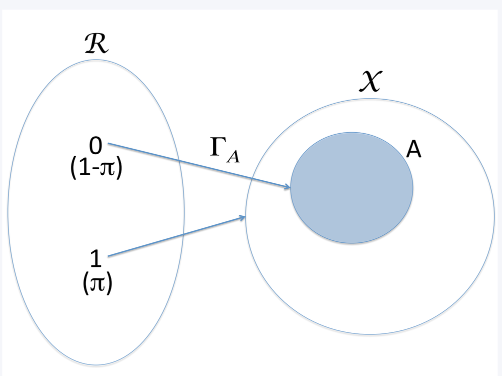
- Only some intuition is provided for the debt of belief semantics given to $A^w$, $w > 1$, it lacks an operation definition.
- Accepting the existence of this notion implies considering a theory richer than the theory of belief functions, which remains to be proposed.

## New Decomposition

### Belief functions for the representation of partially reliable and elementary testimonies

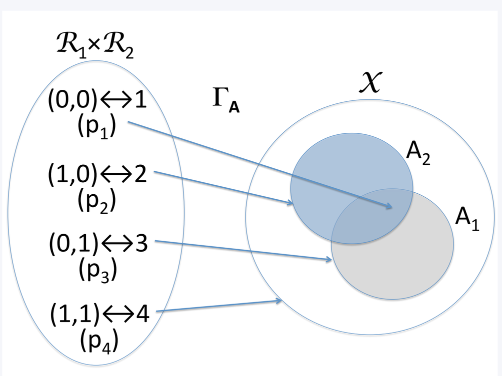
Case of a single source

- Let $\mathfrak{s}$ be a source providing the testimony $x \in A \subseteq \mathcal{X}$:
  - If $\mathfrak{s}$ is reliable, we should deduce $x \in A$;
  - If $\mathfrak{s}$ is not reliable, we know nothing ($x \in \mathcal{X}$).
- Following Dempster [1], this reasoning may be encoded as follows. Let $R$ be the variable denoting the reliability of $\mathfrak{s}$, defined on $\mathcal{R} = \{0, 1\}$ where $0$ means $\mathfrak{s}$ is reliable and $1$ means not reliable. The interpretation of testimony $A$ of $\mathfrak{s}$ according to its reliability may then be represented by $\Gamma_A : \mathcal{R} \to 2^{\mathcal{X}}$ s.t. $\Gamma_A(0) = A, \Gamma_A(1) = \mathcal{X}$.
- If $\mathfrak{s}$ is assumed to be reliable with probability $1 - \pi$, then knowledge induced about $x$ is represented by SSF $A^{\pi}$.




Case of several sources

- Consider now some sources $\mathfrak{s}_i$, $i = 1, \ldots, N$, providing testimonies $\mathbf{A} = (A_1, \ldots, A_N)$.
- Let $\Gamma_{A_i} : \mathcal{R}_i \to 2^{\mathcal{X}}$ represent the interpretation of testimony $A_i$ of $\mathfrak{s}_i$ according to its reliability $R_i$ defined on $\mathcal{R}_i = \{0, 1\}$.
- The interpretation of testimonies $\mathbf{A}$ when the sources are in state $\mathbf{k} = (k_1, \ldots, k_N) \in \times_{i=1}^N \mathcal{R}_i$ is $x \in \Gamma_{\mathbf{A}}(\mathbf{k}) := \cap_{i=1}^N \Gamma_{A_i}(k_i)$.
- $\mathbf{k} \leftrightarrow k := 1 + \sum_{i=1}^N k_i 2^{i-1}$.
- If each state $k$ is allocated probability $p_k$, then the testimonies are interpreted as $m(B) = \sum_{k : \Gamma_{\mathbf{A}}(k) = B} p_k$.




→ Any set of partially reliable and elementary testimonies is represented by a belief function.

***Proposition*** *Let $m$ be a mass function on $\mathcal{X} = \{x_1, \ldots, x_n\}$. If $N = n$, $A_i = \overline{\{x_i\}}$, and $p_k = m(A^k)$ with $A^k$ the $k$-th subset of $\mathcal{X}$ according to the binary order, then the testimonies are interpreted as $m$.*

→ Any belief function represents (at least) a set of partially reliable and elementary testimonies.

### Marginal reliabilities and dependencies between the reliabilities

Knowledge $p_k = P(R_1 = k_1, \ldots, R_n = k_n)$ on the reliability of the sources is a multivariate Bernoulli distribution. Teugels [7] shows that it is characterized by

$$\pi_i = \mathbb{E}[R_i]$$

and

$$\sigma_k = \mathbb{E}\left[\prod_{i=1}^n (R_i - \pi_i)^{k_i}\right].$$

## New Decomposition (cont.)

Any mass function $m$ on $\mathcal{X} = \{x_1, \ldots, x_n\}$ is then induced by the following basic components:

1. Testimonies $x \in \overline{\{x_i\}}$ provided by sources $\mathfrak{s}_i$, $i = 1, \ldots, n$;
2. Knowledge on the reliability of $\mathfrak{s}_i$ in the form of $\pi_i$;
3. Knowledge on the dependency between the reliabilities of the sources $\in S_k = \{\mathfrak{s}_i : i, k_i = 1\}$ in the form of $\sigma_k$.

Besides, we have

$$\pi_i = pl(x_i),$$

$$\sigma_k = \left( \begin{bmatrix} 1 & 0 \\ -q(\{x_n\}) & 1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & 0 \\ -q(\{x_1\}) & 1 \end{bmatrix} \mathbf{q} \right)(A_k). \quad (1)$$

### Presentation in terms of SSF

According to Destercke and Dubois's approach [2], the conjunctive combination $m_{\cap}$ of mass functions $m_1, \ldots, m_n$ under known dependency is obtained by :

1. Building a joint mass function $jm : \times_{i=1}^n 2^{\mathcal{X}} \to [0,1]$ having $m_1, \ldots, m_n$ as marginals and representing their mutual dependencies;
2. Allocating each joint mass $jm(A_1, \ldots, A_n)$ to $\bigcap_{i=1}^n A_i$:

$$m_{\cap}(A) = \sum_{\cap_{i=1}^n A_i = A} jm(A_1, \ldots, A_n).$$

Interestingly, if $m_i = A_i^{\pi_i}$, then the dependency structure in $jm$ is characterized by a $\boldsymbol{\sigma}$. This allows the definition of a conjunctive combination rule for SSF parameterized by $\boldsymbol{\sigma}$ :

$$\circledcirc_{\boldsymbol{\sigma}} (A_1^{\pi_1}, \ldots, A_n^{\pi_n}) := m_{\cap}.$$

***Theorem*** *Any mass function $m$ satisfies*

$$m = \circledcirc_{\boldsymbol{\sigma}} \left( \overline{\{x_1\}}^{pl(x_1)}, \ldots, \overline{\{x_n\}}^{pl(x_n)} \right),$$

*with $\boldsymbol{\sigma}$ obtained from $q$ by (1).*

This decomposition coincides with Smets's when $\boldsymbol{\sigma} = \mathbf{e}_1$. However, they are different in general.

***Example*** *Let $\mathcal{X} = \{x_1, x_2\}$ and $m(\{x_1\}) = m(\{x_2\}) = m(\{x_1, x_2\}) = 1/3$.*

*Decomposition :*

- *$\mathfrak{s}_1$ tells $x \in \overline{\{x_1\}} = \{x_2\}$ and $\mathfrak{s}_2$ tells $x \in \overline{\{x_2\}} = \{x_1\}$;*
- *$\mathfrak{s}_1$ and $\mathfrak{s}_2$ each not reliable with (marginal) probability $2/3$;*
- *Covariance of $-1/9$ between their reliabilities.*

*Presentation in terms of SSF:* $m = \circledcirc_{(-1/9)} \left( \{x_2\}^{2/3}, \{x_1\}^{2/3} \right)$.

*Smets's decomposition:* $m = \{x_2\}^{1/2} \circledcirc \{x_1\}^{1/2} \circledcirc \emptyset^{4/3}$.

## New Perspective on the Weight Function

Smets's decomposition can be equivalently presented using $s(A) = -\ln w(A)$ for all $A \subset \mathcal{X}$ (Shafer's weights).

For $\mathcal{X} = \{x_1, x_2\}$, we have

$$s(\emptyset) = I(R_1 = 1; R_2 = 1),$$

with $I(R_1 = 1; R_2 = 1)$ the mutual information [3] that the sources $\mathfrak{s}_1$ and $\mathfrak{s}_2$ underlying $m$ are not reliable.

→ A completely different meaning for $s(\emptyset) < 0$ than that of debt of belief.

We have also, for instance,

$$s(\{x_1\}) = I(R_2 = 1 | R_1 = 1),$$

with $I(R_2 = 1 | R_1 = 1)$ the conditional self information that $\mathfrak{s}_2$ is not reliable given that $\mathfrak{s}_1$ is not reliable.

This kind of semantics for $s(A)$, $A \subset \mathcal{X}$, is obtained for any cardinality of $\mathcal{X}$.

## Conclusions

Besides the representation of elementary testimonies having independent reliabilities, the theory of belief functions allows also the representation of elementary testimonies having dependent reliabilities. More precisely, whatever the considered set of partially reliable and elementary testimonies (and in particular whatever the dependencies between their reliabilities) there exists a unique belief function representing it, and, importantly, any belief function can be associated uniquely to a particular set of partially reliable and elementary testimonies inducing it.

This new decomposition does not suffer from the criticisms that have been addressed to Shafer and Smets's decompositions. Above all, it casts a fresh light on belief functions that may be useful to tackle several issues. A first example is the weight function underlying Smets's proposal, which instead of interpreting with some difficulty as a decomposition of a belief function into SSF, can be given a different and well-defined semantics in terms of measures of information associated with the reliabilities of the elementary testimonies in the new decomposition.

## References

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