Exposing some points of interest about non-exposed points of desirability

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Context: Sets of desirable gambles on a finite space

Gambles The random variable X takes values in the **finite possibility space** Ω. Any real-valued function on Ω is called a **gamble**, and we collect all of them in $\mathscr{L}$. For any two gambles f and g, we introduce $f \leq g \Leftrightarrow (\forall x \in \Omega) f(x) \leq g(x)$, and $f < g \Leftrightarrow (\forall x \in \Omega) f(x) < g(x)$.

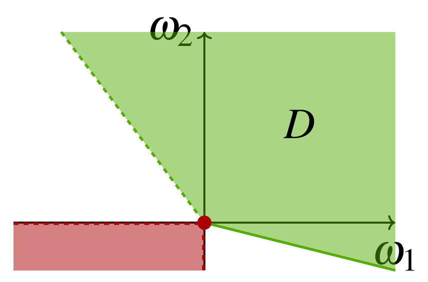
Sets of desirable gambles A subject's beliefs about X are modelled by means of his **set of desirable gambles**, which is a subset D of gambles that he (strictly) prefers to the status quo indicated by 0.

Rationality axioms We call a set of desirable gambles D **coherent** if for all gambles f and g and all real $\lambda > 0$:

D_1. $0 \notin D$; [**avoiding null gain**]
D_2. if $0 < f$ then $f \in D$; [**desiring sure gain**]
D_3. if $f \in D$ then $\lambda f \in D$; [**positive scaling**]
D_4. if $f, g \in D$ then $f + g \in D$. [**combination**]

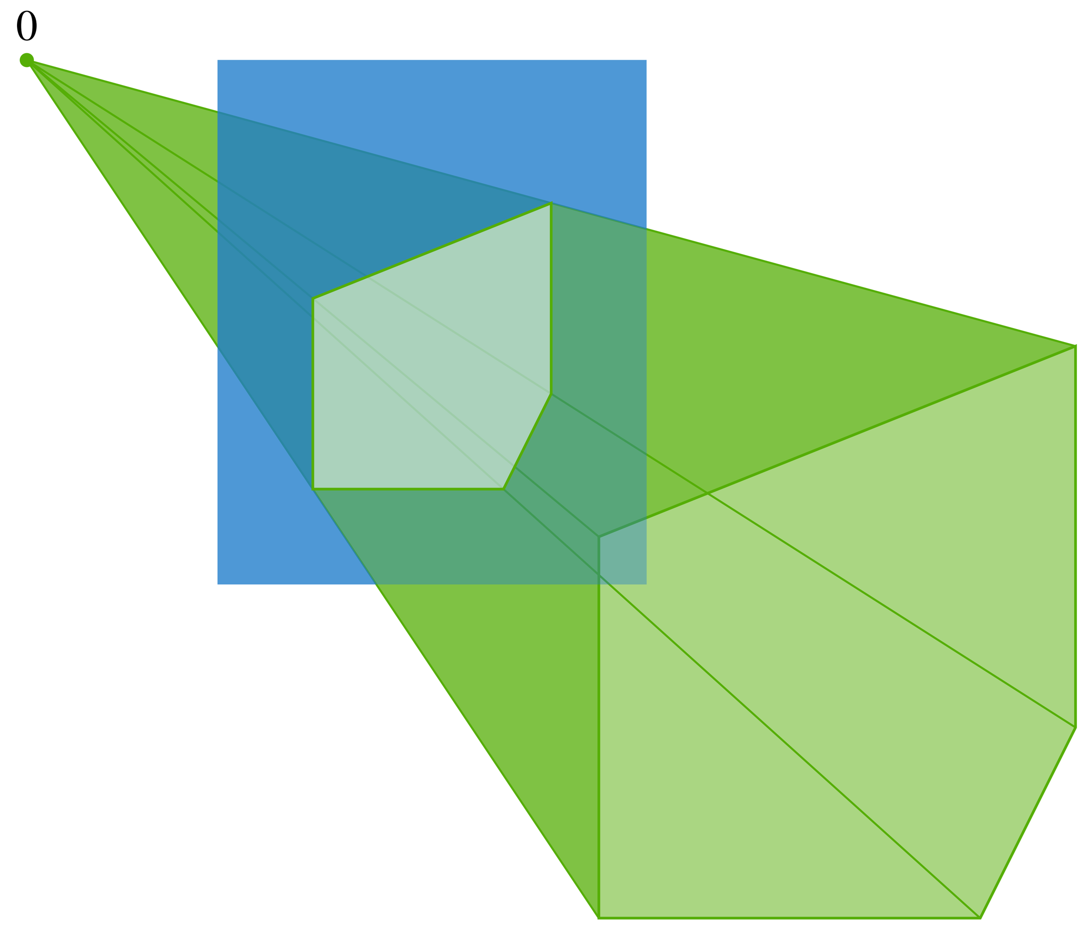
A coherent set of desirable gambles is a convex cone that includes the gambles $f > 0$ and has nothing in common with $\{0\}$ and the gambles $f < 0$. Note that Axiom D_2 only requires that point-wise positive gambles be desirable: we do not require **admissibility** or **weak dominance**, even in finite state spaces.

Binary possibility space $\Omega = \{\omega_1, \omega_2\}$



Full lines are included; dashed lines not.

Ternary possibility space $\Omega = \{\omega_1, \omega_2, \omega_3\}$



Probabilities Given a probability mass function p with corresponding expectation operator E_p, the set

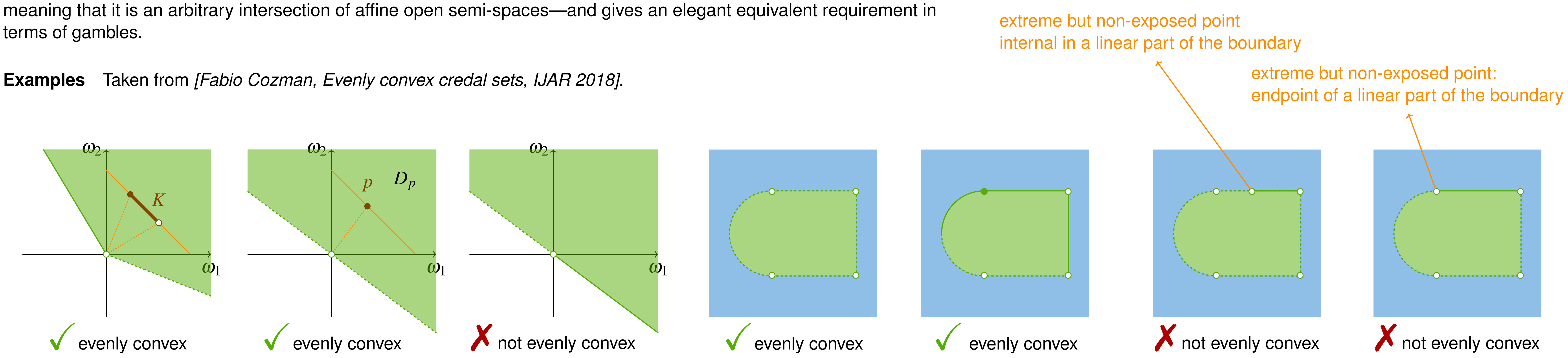
$$D_p := \{f \in \mathscr{L} : E_p(f) > 0\},$$

is coherent. It is the smallest coherent set of desirable gambles whose lower prevision is equal to E_p. We have that every set of desirable gambles D is in fact a D_p if and only if it an open semispace that includes the gambles $f > 0$. We say that a set of desirable gambles D is **represented** by a set K of probability mass functions when $D = \bigcap_{p \in K} D_p$.

When is a coherent set of desirable gambles represented by a set of probability mass functions?

If D is represented by a set of probabilities, then the largest K that represents D is convex, but not necessarily closed. *[Fabio Cozman, Evenly convex credal sets, IJAR 2018]* pointed out that D is represented by K if and only if D is **evenly convex**—meaning that it is an arbitrary intersection of affine open semi-spaces—and gives an elegant equivalent requirement in terms of gambles.

Examples Taken from *[Fabio Cozman, Evenly convex credal sets, IJAR 2018]*.



We will show that even convexity follows also from the two axioms **SSK–Archimedeanity** and **SSK–Extension** from *[Seidenfeld, Schervish, and Kadane, A representation of partially ordered preferences. The Annals of Statistics 1995]*.

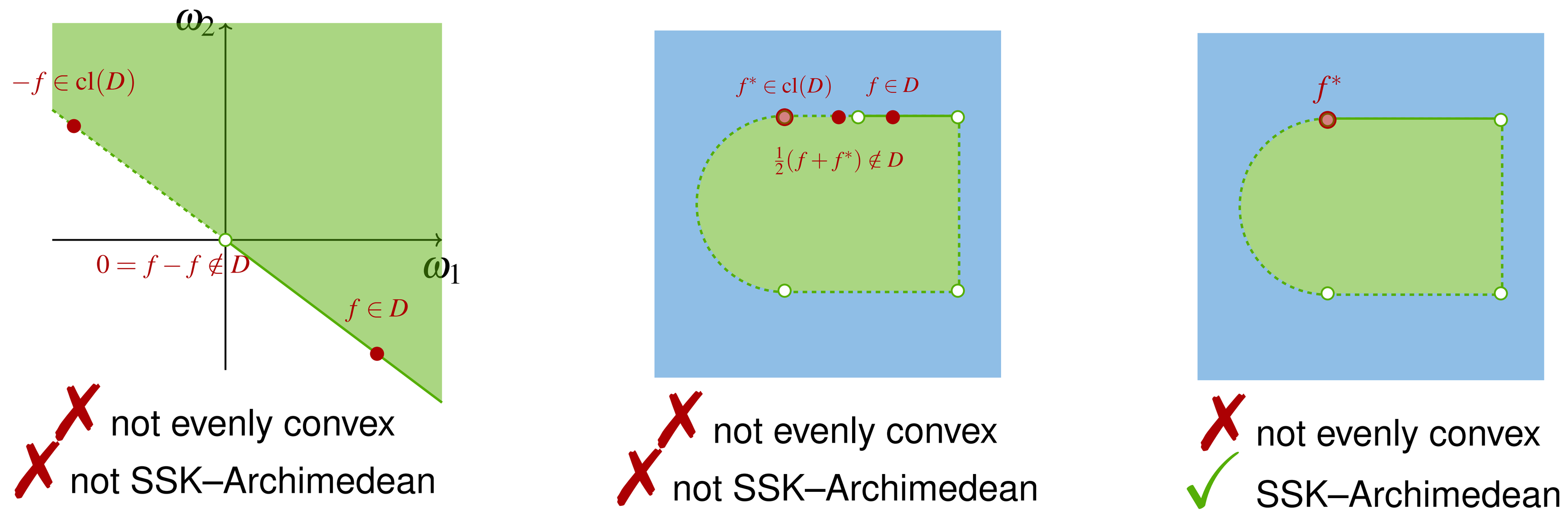
SSK–Archimedeanity

The requirement **SSK–Archimedeanity** from *[Seidenfeld, Schervish, and Kadane, A representation of partially ordered preferences. The Annals of Statistics 1995]*, expressed for gambles, is (with the aid of a lemma) equivalent to:

Definition (SSK–Archimedeanity) A set of desirable gambles is called SSK–Archimedean when $D + \mathrm{cl}(D) \subseteq D$, where for any sets of gambles A and B, their addition is defined as $A + B := \{a + b : a \in A, b \in B\}$.

SSK–Archimedeanity takes care of the internal part of the boundary:

Proposition Consider any coherent set of desirable gambles D. If D is SSK–Archimedean, then for any gamble f on a linear part of the boundary of D, if $f \in D$ then the whole interior of this linear part of the boundary is contained in D.



SSK–Extension

For the requirement **SSK–Extension** from *[Seidenfeld, Schervish, and Kadane, A representation of partially ordered preferences. The Annals of Statistics 1995]*, we need two new notions:

Precluded desirability A gamble f is called **precluded from being desirable** when $-f \in \mathrm{cl}(D)$.

Precluded p-indifference A gamble f is called **precluded from being p-indifferent to 0** if assuming that there is a probability mass function p such that $E_p(f) = 0$ will be incompatible with D, in the sense that $E_p(g) \leq 0$ for some g in D.

Definition (SSK–Extension) For any coherent set of desirable gambles D, its SSK–Extension D^* is given by

$D^* := \{f \in \mathscr{L} : f \in D$ or
$(-f$ is precluded from being desirable and f is precluded from being p-indifferent to $0)\}$.

SSK–Archimedeanity and SSK–Extension take care of the boundary:

Proposition Consider any coherent set of desirable gambles D that is SSK–Archimedean. Then D^* is coherent and evenly convex.

Ice Cream Cone Corollary Consider any coherent set of desirable gambles D that is SSK–Archimedean. Then, D contains all its extreme non-exposed points if and only if D is evenly convex. This extends Theorem 16 of *[Fabio Cozman, Evenly convex credal sets, IJAR 2018]*.

"IP Ice Cream Cone Theorem"