Belief models

A very general theory of aggregation

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Belief models [2] offer a very general theory of rational belief that generalises propositional logic and lower previsions (among others). Tools from propositional logic can be "imported" into the BM framework, and then applied to other instances of BM (e.g. lower previsions) in order to generate new results. We use this basic idea to import the theory of "merging operators" into the lower prevision framework, and use this to generate aggregation rules for IP that are guaranteed to have certain desirable properties.

Belief models

Let $\mathbf{S}$ be a set of *belief models*, partially ordered by $\preceq$ (read as "is less informative than"), such that $\langle \mathbf{S}, \preceq \rangle$ is a complete lattice. Let $\mathbf{C} \subseteq \mathbf{S}$ be the subset of *coherent* belief models, and stipulate that $\mathbf{C}$ is closed under arbitrary non-empty infima. In particular, $1_{\mathbf{S}} \notin \mathbf{C}$. $\langle \mathbf{S}, \mathbf{C}, \preceq \rangle$ is called a *belief structure*. Let $\mathbf{M} = \{m \in \mathbf{C} : \text{For all } c \in \mathbf{C}, m \preceq c \Rightarrow m = c\}$ Call a belief structure a *strong* belief structure, when, for all $c \in \mathbf{C}$, $c = \inf\{m \in \mathbf{M}, c \preceq m\}$. Later we will also appeal to the following property:

$$\text{For distinct } a, b, c \in \mathbf{M}, c \npreceq a \wedge b \qquad (*)$$

Examples of belief structures include:

- Propositional logic (with $\subseteq$, and consistent sets closed under consequence)
- Lower previsions (with pointwise dominance and closed convex credal sets)
- Modal logics and other nonstandard logics with well-behaved consequence operator
- Ranking functions
- Sets of desirable gambles, choice functions...
- Preference relations, comparative confidence relations?

AGM in BM

Call K_A^+ the expansion of K by (consistent) A.

1. K_A^+ is a belief set (i.e. closed under entailment and consistent)
2. $A \in K_A^+$
3. $K \subseteq K_A^+$
4. If $A \in K$ then $K_A^+ = K$
5. If $K \subseteq H$ then $K_A^+ \subseteq H_A^+$
6. For all K and A, K_A^+ is the smallest belief set satisfying the above conditions

If K_A^+ satisfies the above conditions, then $K_A^+ = Cn(K \cup \{A\})$.

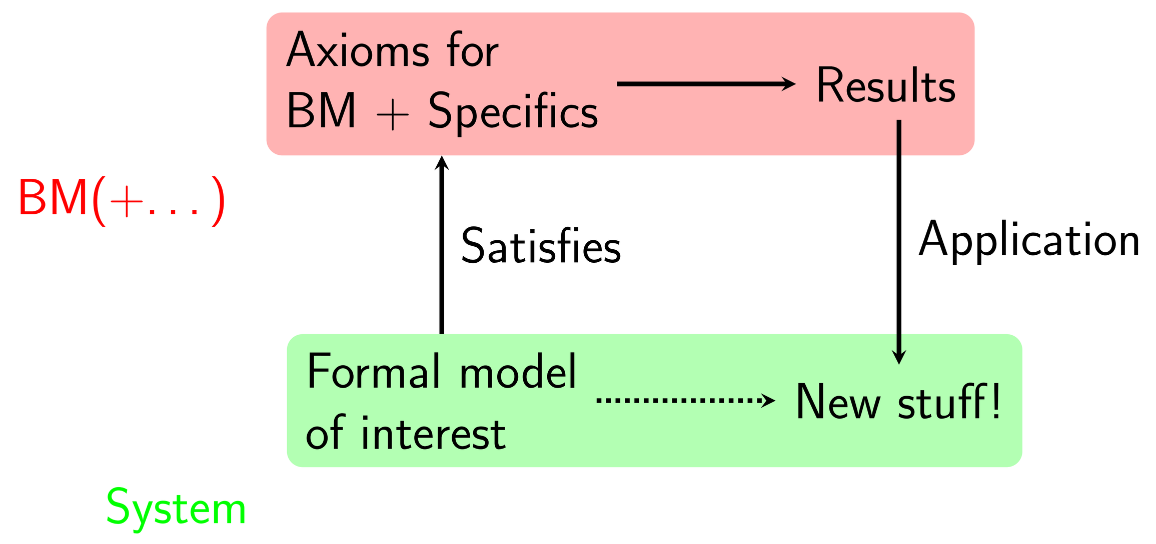
Call $E(b, c)$ the expansion operator for learning c on having beliefs b.

1. $E(b, c) \in \overline{\mathbf{C}}$
2. $c \preceq E(b, c)$
3. $b \preceq E(b, c)$
4. If $c \preceq b$ then $E(b, c) = b$
5. If $b \preceq d$ then $E(b, c) \preceq E(d, c)$
6. $E(b, -)$ is the least informative of all the operators satisfying the above

If E satisfies the above, then $E(b, c) = Cl_{\mathbf{S}}(\sup\{b, c\})$.

If the belief structure is strong, then the same sort of thing can be done for AGM revision.

The goal is to do the same sort of thing that de Cooman does for AGM, but for the "merging operators" literature [1, 3, 4, 5].



Merging operators for BMs

Call $\Delta(\Psi, \mu)$ – or $\Delta_\mu(\Psi)$ – a *merging operator* if Ψ is a multiset of belief models, and μ is a belief model representing the constraints the aggregate belief must satisfy, and Δ satisfies:

- $\mu \preceq \Delta_\mu(\Psi)$
- If μ is consistent then $\Delta_\mu(\Psi)$ is consistent
- If $\bigvee \Psi \vee \mu$ is consistent then $\Delta_\mu(\Psi) = \bigvee \Psi \vee \mu$
- If $\mu \preceq \phi_1$ and $\mu \preceq \phi_2$ then $\Delta_\mu(\phi_1 \sqcup \phi_2) \vee \phi_1$ is consistent if and only if $\Delta_\mu(\phi_1 \sqcup \phi_2) \vee \phi_2$
- $\Delta_\mu(\Psi_1 \sqcup \Psi_2) \preceq \Delta_\mu(\Psi_1) \vee \Delta_\mu(\Psi_2)$
- If $\Delta_\mu(\Psi) \vee \Delta_\mu(\Psi_2)$ is consistent then, $\Delta_\mu(\Psi_1) \vee \Delta_\mu(\Psi_2) \preceq \Delta_\mu(\Psi_1 \sqcup \Psi_2)$
- $\Delta_{\mu_1 \vee \mu_2}(\psi) \preceq \Delta_{\mu_1}(\Psi) \vee \mu_2$
- If $\Delta_{\mu_1}(\Psi) \vee \mu_2$ is consistent then $\Delta_{\mu_1}(\Psi) \vee \mu_2 \preceq \Delta_{\mu_1 \vee \mu_2}(\psi)$

The main results (inspired by [3, 5]) are:

- If Δ is a merging operator, then define $K_\mu^* = \Delta_\mu(K)$. This is AGM revision.
- Every merging operator (satisfying some properties) yields an entrenchment relation over the maximal coherent elements (...), and vice versa.
- Every "distance" and method of aggregating distances (...) yields a merging operator (...)

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Distance-based merging

One approach to constructing merging operators is to start from a distance between maximal belief models: $D(m, m')$. Define a distance between worlds and belief sets:

$$D(m, \phi) = \min_{\phi \preceq m'}\{D(m, m')\}$$

Define a distance between worlds and multisets of belief sets:

$$D(m, \Psi) = \sum_{\phi \in \Psi} D(m, \phi)$$

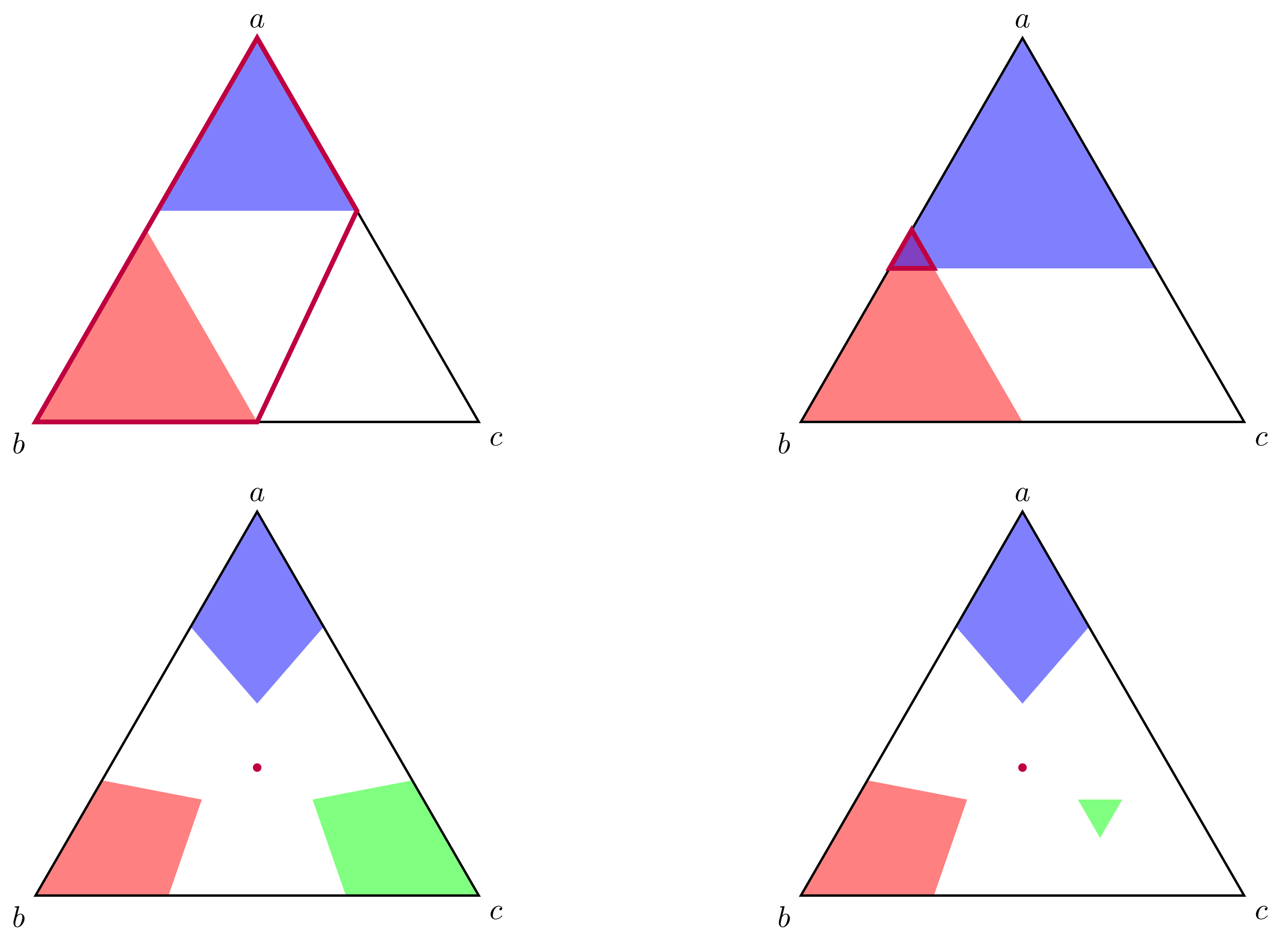
Then aggregate by minimising that distance.
For example: start with the so-called "drastic distance":

$$D_d(m, m') = \begin{cases} 0 \text{ if } m = m' \\ 1 \text{ otherwise} \end{cases}$$

$$D_d(m, \phi) = \min_{\phi \preceq m'}\{D_d(m, m')\} = \begin{cases} 0 \text{ if } m \in M(\phi) \\ 1 \text{ otherwise} \end{cases}$$

$$D_d(m, \Psi) = \sum D_d(m, \phi) = \text{The number of } \phi \in \Psi \text{ that } m \text{ is not in.}$$

Then we minimise that: meaning, we pick the maximal (w.r.t cardinality) consistent subsets.



Distance-based merging axioms

Distance:

- D maps pairs of maximal coherent belief models to real numbers
- $D(m, m') = D(m', m)$
- $D(m, m') = 0$ iff $m = m'$.

Aggregation:

- F takes a sequence of real numbers and outputs a real number
- If $x \leq y$ then $F(x_1, \ldots, x, \ldots, x_n) \leq F(x_1, \ldots, y, \ldots, x_n)$
- $F(x_1, \ldots, x_n) = 0$ iff $x_1 = \cdots = x_n = 0$
- For all $x \in \mathbb{R}$, $F(x) = x$
- For a permutation σ, $F(x_1, \ldots, x_n) = F(\sigma(x_1), \ldots, \sigma(x_n))$
- $F(x_1, \ldots, x_n) \leq F(y_1, \ldots, y_n) \Rightarrow F(x_1, \ldots, x_n, z) \leq F(y_1, \ldots, y_n, z)$
- $F(x_1, \ldots, x_n) \leq F(y_1, \ldots, y_n) \Leftarrow F(x_1, \ldots, x_n, z) \leq F(y_1, \ldots, y_n, z)$

- What about impossibility theorems?
- How weak is the additional property? Can we weaken "strongness" to something involving infima of maximal ideals?
- Are there any interesting strong BMs that are not distributive?
- Can we recover Dempster-Shafer combination in this framework?
- How does this relate to the work on fusion of possibilistic knowledge bases (also inspired by Konienczny and Pino Pérez)?
- Which forms of aggregation commute with updating?

References

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