# Semi-Graphoid Properties of Variants of Epistemic Independence based on Regular Conditioning

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## Goal:

to study the semi-graphoid properties of concepts of independence based on regular conditioning.

## A bit of background...

- Credal set: a set of (Kolmogorovian-style) probability measures.
  - Focus on finite spaces.
  - Credal sets may be open, may fail to be convex.
- Graphoid properties:

  Symmetry: $(X \perp\!\!\!\perp Y \mid Z) \Rightarrow (Y \perp\!\!\!\perp X \mid Z)$

  Redundancy: $(X \perp\!\!\!\perp Y \mid X)$

  Decomposition: $(X \perp\!\!\!\perp (W, Y) \mid Z) \Rightarrow (X \perp\!\!\!\perp Y \mid Z)$

  Weak union:

  $(X \perp\!\!\!\perp (W, Y) \mid Z) \Rightarrow (X \perp\!\!\!\perp Y \mid (W, Z))$

  Contraction: $(X \perp\!\!\!\perp Y \mid Z) \wedge (X \perp\!\!\!\perp W \mid (Y, Z)) \Rightarrow (X \perp\!\!\!\perp (W, Y) \mid Z)$

  Intersection

  $(X \perp\!\!\!\perp W \mid (Y, Z))$ & $(X \perp\!\!\!\perp Y \mid (W, Z)) \Rightarrow (X \perp\!\!\!\perp (W, Y) \mid Z)$.
- Semi-graphoid properties: all of them except Intersection.
- Regular conditioning:

  $\mathbb{K}^{>}(X|H) = \{\mathbb{P}(\cdot|H) : \mathbb{P} \in \mathbb{K}(X) \text{ and } \mathbb{P}(H) > 0\}$ whenever $\overline{\mathbb{P}}(H) > 0$.

## A menu of independences

- $Y$ is *regular-confirmationally irrelevant* to $X$ given $Z$:

  $\mathbb{K}^{>}(X|y, z) = \mathbb{K}^{>}(X|z)$ whenever $\overline{\mathbb{P}}(y, z) > 0$.
- $Y$ is *regular-epistemically irrelevant* to $X$ given $Z$:

  $\underline{\mathbb{E}}^{>}[f|y, z] = \underline{\mathbb{E}}^{>}[f|z]$ whenever $\overline{\mathbb{P}}(y, z) > 0$.
- $Y$ is *type-5 irrelevant* to $X$ given $Z$:

  $\mathbb{K}^{>}(X|B, z) = \mathbb{K}^{>}(X|z)$ whenever $\overline{\mathbb{P}}(B, z) > 0$.
- $Y$ is *type-5 epistemically irrelevant* to $X$ given $Z$:

  $\underline{\mathbb{E}}^{>}[f|B, z] = \underline{\mathbb{E}}^{>}[f|z]$ whenever $\overline{\mathbb{P}}(B, z) > 0$.
- All of them fail Symmetry.
- By "symmetrizing" we get: regular-confirmational, regular-epistemic, type-5 type-5 epistemic *independence*.

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## The results

### Theorem

*If $(Y \,\mathrm{IR}\, X \mid Z)$ denotes regular-epistemic irrelevance of $Y$ to $X$ given $Z$, then:*

- *$(X \,\mathrm{IR}\, Y \mid X)$ and $(Y \,\mathrm{IR}\, X \mid X)$;*
- *If $(X \,\mathrm{IR}\, W, Y \mid Z)$, then $(X \,\mathrm{IR}\, Y \mid Z)$;*
- *If $(X \,\mathrm{IR}\, W, Y \mid Z)$, then $(X \,\mathrm{IR}\, Y \mid W, Z)$ [NOTE: FAILS FOR de Finettian-conditioning!];*
- *If $(Y \,\mathrm{IR}\, X \mid Z)$ and $(W \,\mathrm{IR}\, X \mid Y, Z)$, then $(W, Y \,\mathrm{IR}\, X \mid Z)$.*

### Theorem

*If $(Y \,\mathrm{IR}\, X \mid Z)$ denotes regular-confirmational irrelevance of $Y$ to $X$ given $Z$, then the same properties listed in the previous theorem hold for $\mathrm{IR}$.*

### Theorem

*Regular-confirmational and regular-epistemic independence satisfy Symmetry and Redundancy. (And fail all other properties! They satisfy Decomposition and Weak Union when lower probabilities are larger than zero.)*

### Theorem

*If $(Y \,\mathrm{IR}\, X \mid Z)$ denotes type-5 epistemic irrelevance of $Y$ to $X$ given $Z$, then:*

- *$(X \,\mathrm{IR}\, Y \mid X)$ and $(Y \,\mathrm{IR}\, X \mid X)$;*
- *If $(X \,\mathrm{IR}\, W, Y \mid Z)$, then $(X \,\mathrm{IR}\, Y \mid Z)$;*
- *If $(X \,\mathrm{IR}\, W, Y \mid Z)$, then $(X \,\mathrm{IR}\, Y \mid W, Z)$;*
- *If $(W, Y \,\mathrm{IR}\, X \mid Z)$, then $(Y \,\mathrm{IR}\, X \mid Z)$;*
- *If $(W, Y \,\mathrm{IR}\, X \mid Z)$, then $(Y \,\mathrm{IR}\, X \mid W, Z)$;*

### Theorem

*If $(Y \,\mathrm{IR}\, X \mid Z)$ denotes type-5 irrelevance of $Y$ to $X$ given $Z$, then the same properties listed in the previous theorem hold.*

### Theorem

*Type-5 independence and type-5 epistemic independence both satisfy Symmetry, Redundancy, Decomposition and Weak Union.*

## Complete and strong independence

- Complete independence satisfies all semi-graphoid properties.
- Strong independence satisfies Symmetry, Redundancy, Decomposition and Weak Union — but it fails Contraction!

## Conclusion

- In this paper: a detailed map of semi-graphoid properties (all properties not mentioned fail...!).
- Confirmational/epistemic seem very weak... "type-5 condition" leads to better behavior.