Interpreting, Axiomatising and Representing Coherent Choice Functions in Terms of Desirability

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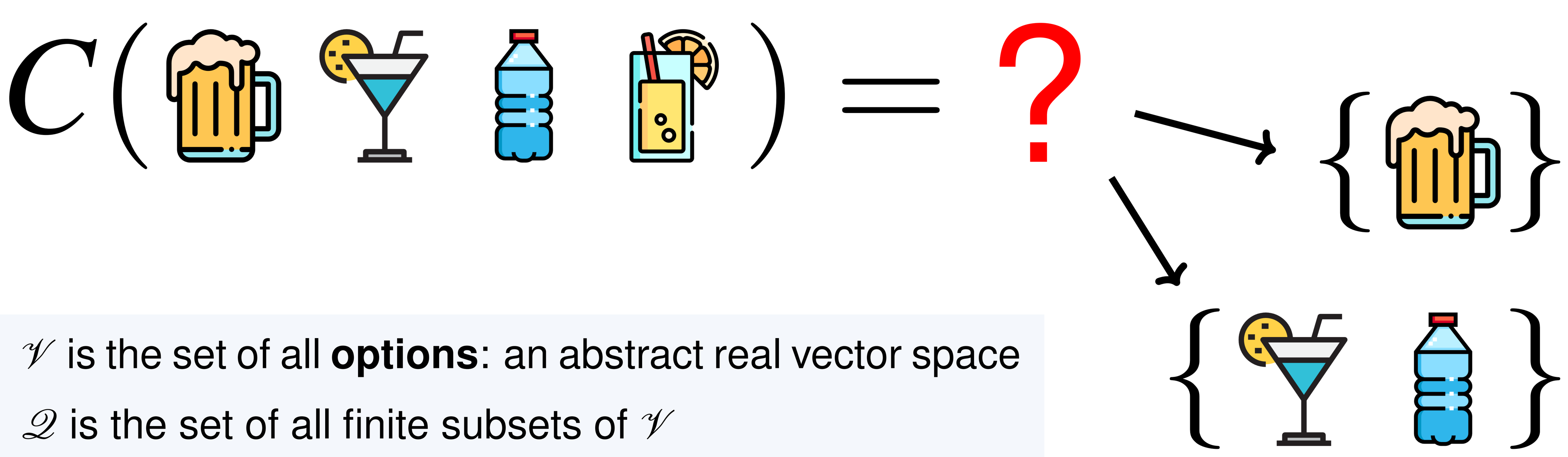

a unifying framework for modelling set-valued choice!

$\mathscr{V}$ is the set of all **options**: an abstract real vector space

$\mathscr{Q}$ is the set of all finite subsets of $\mathscr{V}$

a **choice function** C is a map from $\mathscr{Q}$ to $\mathscr{Q}$ such that $C(A) \subseteq A$ for every $A \in \mathscr{Q}$

the corresponding **rejection function** R is defined by $R(A) := A \setminus C(A)$, for all $A \in \mathscr{Q}$

desirability provides an interpretation for each of our models

	D	K	R / C
	an option $v \in \mathscr{V}$ is desirable if it is strictly preferred to zero	an option set $A \in \mathscr{Q}$ is desirable if it contains at least one desirable option	an option $u \in A$ is rejected from A if at least one option $v \in A$ is strictly preferred over u, in the sense that $v - u$ is desirable
	Set of desirable options	Set of desirable options sets	Rejection function / Choice function
COHERENT	D_1 $0 \notin D$ D_2 $\mathscr{V}_{\succ 0} \subseteq D$ (1) D_3 if $u, v \in D$ and $(\lambda, \mu) > 0$, (2) then $\lambda u + \mu v \in D$	K_0 if $A \in K$ then also $A \setminus \{0\} \in K$, for all $A \in \mathscr{Q}$ K_1 $\{0\} \notin K$ K_2 $\{u\} \in K$, for all $u \in \mathscr{V}_{\succ 0}$ K_3 if $A_1, A_2 \in K$ and if, for all $u \in A_1$ and $v \in A_2$ $(\lambda_{u,v}, \mu_{u,v}) > 0$, then also $\{\lambda_{u,v} u + \mu_{u,v} v \colon u \in A_1, v \in A_2\} \in K$ K_4 if $A_1 \in K$ and $A_1 \subseteq A_2 \in \mathscr{Q}$, then also $A_2 \in K$	R_0 for all $A \in \mathscr{Q}$ and $u \in A$: $u \in R(A) \Leftrightarrow 0 \in R(A - u)$ R_1 $R(\emptyset) = \emptyset$, and $R(A) \neq A$ for all $A \in \mathscr{Q} \setminus \{\emptyset\}$ R_2 $0 \in R(\{0, u\})$, for all $u \in \mathscr{V}_{\succ 0}$ R_3 if $A_1, A_2 \in \mathscr{Q}$, $0 \in R(A_1 \cup \{0\})$ and $0 \in R(A_2 \cup \{0\})$ and if $(\lambda_{u,v}, \mu_{u,v}) > 0$ for all $u \in A_1$ and $v \in A_2$, then $0 \in R(\{\lambda_{u,v} u + \mu_{u,v} v \colon u \in A_1, v \in A_2\} \cup \{0\})$ R_4 if $A_1 \subseteq A_2$ then $R(A_1) \subseteq R(A_2)$, for all $A_1, A_2 \in \mathscr{Q}$
TOTAL	D_T for all $u \in \mathscr{V} \setminus \{0\}$, either $u \in D$ or $-u \in D$	K_T $\{u, -u\} \in K$ for all $u \in \mathscr{V} \setminus \{0\}$	R_T $0 \in R(\{0, u, -u\})$, for all $u \in \mathscr{V} \setminus \{0\}$
MIXING	D_M if $\mathrm{posi}(A) \cap D \neq \emptyset$, then also $A \cap D \neq \emptyset$, for all $A \in \mathscr{Q}$ (3)	K_M if $B \in K$ and $A \subseteq B \subseteq \mathrm{posi}(A)$, then also $A \in K$, for all $A, B \in \mathscr{Q}$	R_M if $A \subseteq B \subseteq \mathrm{conv}(A)$ then also $R(B) \cap A \subseteq R(A)$, for all $A, B \in \mathscr{Q}$ (4)
ARCHIMEDEAN (5)	D_A for all $u \in D$, there is an $\varepsilon \in \mathbb{R}_{>0}$ such that $u - \varepsilon \in D$	K_A for all $A \in K$, there is an $\varepsilon \in \mathbb{R}_{>0}$ such that $A - \varepsilon \in K$	R_A for all $A \in \mathscr{Q}$ and $u \in \mathscr{V}$ such that $u \in R(A \cup \{u\})$, there is some $\varepsilon \in \mathbb{R}_{>0}$ such that $u \in R((A - \varepsilon) \cup \{u\})$
OTHER PROPERTIES?			

$$K_D := \{A \in \mathscr{Q} \colon A \cap D \neq \emptyset\}$$

$$D_K := \{u \in \mathscr{V} \colon \{u\} \in K\}$$

loss of information! (unless binary)

$$\{v - u \colon v \in A\}$$
$$u \in R(A \cup \{u\}) \Leftrightarrow A - u \in K$$

an **intersection** of sets of desirable option sets K amounts to taking the **union** of the corresponding choice functions C

$$K \text{ is } \star \Leftrightarrow K = \cap\{K_D \colon D \in \mathscr{D}\} \quad (6)$$

for some non-empty set $\mathscr{D}$ of $\star$ sets of desirable options

$C \Leftrightarrow$	COHERENT	COHERENT TOTAL	COHERENT MIXING	COHERENT ARCHIMEDEAN	COHERENT MIXING ARCHIMEDEAN
	set of strict (partial) preference orders	set of strict total preference orders	set of lexicographic preference orders	closed set of lower previsions / lower expectations / credal sets	closed set of linear previsions / probability measures

(1) $\mathscr{V}_{\succ 0}$ is a convex cone in $\mathscr{V} \setminus \{0\}$ whose elements must be desirable

(2) $\lambda \geq 0$, $\mu \geq 0$ and $\lambda + \mu > 0$

(3) $\mathrm{posi}(A) := \left\{ \sum_{k=1}^{n} \lambda_k u_k \colon n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, u_k \in A \right\}$

(4) $\mathrm{conv}(A) := \left\{ \sum_{k=1}^{n} \lambda_k u_k \colon n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, \sum_{k=1}^{n} \lambda_k = 1, u_k \in A \right\}$

(5) (so far) only for $\mathscr{V} = \mathscr{L}(\mathscr{X})$: the set of all bounded real-valued functions (gambles) on some set $\mathscr{X}$; $\mathscr{V}_{\succ 0} = \{u \in \mathscr{L}(\mathscr{X}) \colon \inf u > 0\}$

(6) $\mathscr{D}$ is closed in the ARCHIMEDEAN cases; how should we modify Archimedeanity for $\mathscr{D}$ to not be closed?