A Unifying Frame for Neighbourhood and Distortion Models

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Distortion models

Distortion model induced by P_0, d, δ

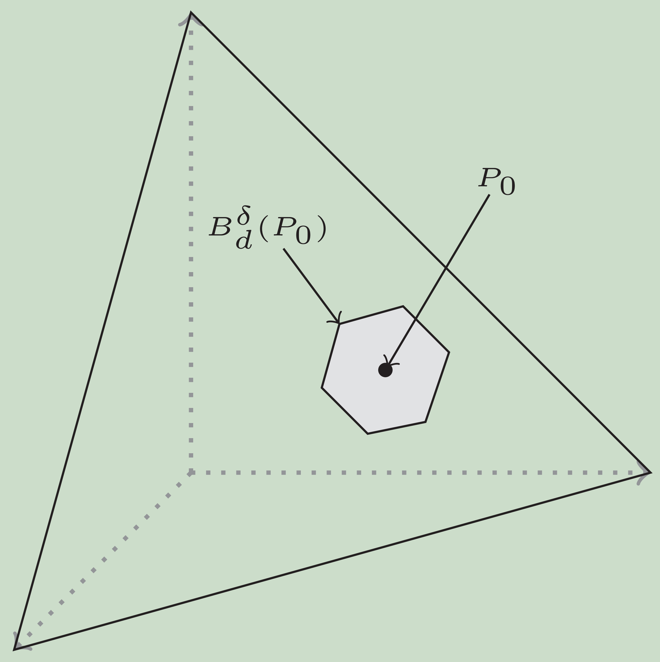
Consider P_0 such that $P_0(A) > 0 \; \forall A \neq \emptyset$.

P_0, d, δ $\longrightarrow B_d^\delta(P_0) = \{P \mid d(P,P_0) \leq \delta\}$

If d is continuous and convex:

- $B_d^\delta(P_0)$ is closed and convex.
- Its lower envelope is a coherent lower prevision.
- This lower prevision is a probability interval if and only if: $d(P,Q) = \max_{x\in\mathcal{X}} \min\{d(P',Q) \mid P'(\{x\}) = P(\{x\})\}$

Example of $B_d^\delta(P_0)$



Transformations of probabilities

If $f : [0,1] \to [0,1]$ is increasing and satisfies

$$f(0) = 0,\; f(1) = 1,\; f(t) \leq t,$$

then:

- $\underline{P} = f(P_0)$ is a monotone lower probability.
- If f is convex, $\underline{P} = f(P_0)$ is 2-monotone.

Proposition: Let $f : [0,1] \to [0,1]$ be an increasing function satisfying $f(0) = 0, f(1) = 1$ and $f(t) \leq t \forall t$. Denote $\underline{P} = f(P_0)$ and let $\mathcal{M}(\underline{P})$ be its credal set. Then there exist a premetric d and $\delta > 0$ such that $\mathcal{M}(\underline{P}) = B_d^\delta(P_0)$.

Examples of distortion models

Central: $B_d^\delta(P_0)$

Distortion models induced by a distance

Total variation distance

$d_{TV}(P,P_0) = \max_A |P(A) - P_0(A)|$

$\underline{P}_{TV}(A) = \max\{0, P_0(A) - \delta\}$

Kolmogorov distance

$d_{TV}(P,P_0) = \max_{x\in\mathcal{X}} |F_P(x) - F_{P_0}(x)|$

$\underline{P}_K(A) = \inf\{P(A) \mid F_P \in (\underline{F}_K, \overline{F}_K)\}$, where:

$\underline{F}_K(x) = F_{P_0}(x) - \delta, \quad \overline{F}_K(x) = F_{P_0}(x) + \delta$

L_1-distance

$d_{L_1}(P,P_0) = \sum_A |P(A) - P_0(A)|$

$\underline{P}_1(A) = P_0(A) - \frac{\delta}{\varphi(n,|A|)}$, where:

$\varphi(n,k) = \sum_{l=0}^{k} \binom{k}{l} \sum_{j=0}^{n-k} \binom{n-k}{j} \left| \frac{l}{k} - \frac{j}{n-k} \right|$

Imprecise Probability models as distortions

Pari mutuel model

$\underline{P}_{PMM}(A) = \max\{(1+\delta)P_0(A) - \delta, 0\} \longrightarrow \mathcal{M}(\underline{P}_{PMM})$

$d_{PMM}(P,P_0) = \max_{A\subseteq\mathcal{X}} \frac{P_0(A)-P(A)}{1-P_0(A)} \longrightarrow B_{d_{PMM}}^\delta(P_0)$

$\mathcal{M}(\underline{P}_{PMM}) = B_{d_{PMM}}^\delta(P_0)$

Linear vacuous mixture

$\underline{P}_{LV}(A) = (1-\delta)P_0(A) \longrightarrow \mathcal{M}(\underline{P}_{LV})$

$d_{LV}(P,P_0) = \max_{A\neq\emptyset} \frac{P_0(A)-P(A)}{P_0(A)} \longrightarrow B_{d_{LV}}^\delta(P_0)$

$\mathcal{M}(\underline{P}_{LV}) = B_{d_{LV}}^\delta(P_0)$

Constant odds ratio

$\mathcal{M}(\underline{P}_{COR}) = \left\{P \mid \frac{P(A)}{P(B)} \geq (1-\delta)\frac{P_0(A)}{P_0(B)} \; \forall A, B \neq \emptyset\right\}$

$d_{COR}(P,P_0) = \max_{A,B\neq\emptyset} \left\{1 - \frac{P(A)\cdot P_0(B)}{P(B)\cdot P_0(A)}\right\} \longrightarrow B_{d_{COR}}^\delta(P_0)$

$\mathcal{M}(\underline{P}_{COR}) = B_{d_{COR}}^\delta(P_0)$

Constant odds ratio (restricted to events)

$\underline{Q}_{COR}(A) = \frac{(1-\delta)P_0(A)}{1-\delta P_0(A)} \longrightarrow \mathcal{M}(\underline{Q}_{COR})$

$d'_{COR}(P,P_0) = \max_{A\neq\mathcal{X},\emptyset} \left\{1 - \frac{P(A)\cdot P_0(A^c)}{P(A^c)\cdot P_0(A)}\right\} \longrightarrow B_{d'_{COR}}^\delta(P_0)$

$\mathcal{M}(\underline{Q}_{COR}) = B_{d'_{COR}}^\delta(P_0)$

Comparative analysis of the distortion models

Amount of imprecision

$B_{d_K}^\delta(P_0) \supseteq B_{d_{TV}}^\delta(P_0)$

$B_{d_{TV}}^\delta(P_0) \subseteq B_{d_{PMM}}^\delta(P_0)$, $B_{d_{TV}}^\delta(P_0) \supseteq B_{d_{LV}}^\delta(P_0)$, $B_{d_{TV}}^\delta(P_0) \subseteq B_{d_1}^\delta(P_0)$

$B_{d_{PMM}}^\delta(P_0) \supseteq B_{d'_{COR}}^\delta(P_0)$, $B_{d_{LV}}^\delta(P_0) \supseteq B_{d'_{COR}}^\delta(P_0)$

$B_{d'_{COR}}^\delta(P_0) \supseteq B_{d_{COR}}^\delta(P_0)$

Properties of the lower probability

Model	2-monotone	Complete monotone	Probability interval	Extreme points of $B_d^\delta(P_0)$
$\underline{P}_{PMM}$	YES	NO	YES	$\frac{n!}{\lfloor\frac{n}{2}\rfloor!(\lfloor\frac{n}{2}\rfloor-1)!(n-\lfloor\frac{n}{2}\rfloor-1)!}$
$\underline{P}_{LV}$	YES	YES	YES	n
$\underline{P}_{TV}$	YES	NO	NO	$\frac{n!}{(\lfloor\frac{n}{2}\rfloor-1)!(n-\lfloor\frac{n}{2}\rfloor-1)!}$
$\underline{P}_{COR}$	NO	NO	NO	$2^n - 2$
$\underline{Q}_{COR}$	YES	YES	NO	$n!$
$\underline{P}_K$	YES	YES	NO	Pell-number $\mathcal{P}_n$
$\underline{P}_1$	NO	NO	NO	Open Problem

Properties of d

d	Properties
d_{PMM}	Premetric (fails symmetry and triangular inequality)
d_{LV}	Premetric (fails symmetry)
d_{COR}	Metric
d'_{COR}	Metric
d_{TV}	Metric
d_K	Metric
d_1	Metric

Conclusions and references

At a glance

- **Distortion models:** distorting a probability by means of a function or a neighbourhood.
- We have compared seven examples by means of the imprecision they introduce, the simplicity of its credal set (in number of extreme points) and the properties of its lower probability.
- No model is uniformly the best, but the LV seems to have the best properties.

References

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