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Markov Chains under Nonlinear Expectation

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Introduction

We consider continuous-time Markov chains with a finite state space of cardinality $d \in \mathbb{N}$. A matrix $q \in \mathbb{R}^{d\times d}$ is called a *Q-matrix* or *transition rate matrix* if it satisfies the following conditions:

- $q_{ii} \le 0$ for all $i = 1, \dots, d$,
- $q_{ij} \ge 0$ for all $i \ne j$,
- $\sum_{j=1}^{d} q_{ij} = 0$ for all $i = 1, \dots, d$.

It is well known that every continuous-time Markov chain with finite state space and certain regularity properties at time $t = 0$ can be related to a Q-matrix and vice versa. More precisely, a matrix $q \in \mathbb{R}^{d\times d}$ is a Q-matrix if and only if it is the generator of a continuous-time Markov chain. In this case, for each $u_0 \in \mathbb{R}^d$, the function

$$u\colon [0,\infty) \to \mathbb{R}^d,\ t \mapsto \mathbb{E}(u_0(X_t))$$

is the unique classical solution to the initial value problem

$$u'(t) = qu(t) \ \text{ for all } t \ge 0, \quad u(0) = u_0.$$

Moreover, it can be shown that a matrix $q \in \mathbb{R}^{d\times d}$ is a Q-matrix if and only if it satisfies the positive maximum principle and $q1 = 0$, where we use the notation $1 = (1, \dots, 1)^T \in \mathbb{R}^d$.

In this paper, we consider continuous-time Markov chains under convex expectations and extend the above relation between Markov chains, Q-matrices and ordinary differential equations to the convex case. This relation is established using convex duality, Nisio semigroups (cf. Nisio (1976/77)) and a convex version of Kolmogorov's extension theorem, see Denk, Kupper, and Nendel (2018). A similar approach has been used by Denk, Kupper, and Nendel (2017) in order to construct Lévy processes under nonlinear expectations via solutions to fully nonlinear PDEs using Nisio semigroups.

The setup

Definition 1. A (possibly nonlinear) map $Q\colon \mathbb{R}^d \to \mathbb{R}^d$ is called a *Q-operator* if the following conditions are satisfied:

- $(Q\lambda e_i)_i \le 0$ for all $\lambda > 0$ and all $i = 1, \dots, d$,
- $(Q(-\lambda e_j))_i \le 0$ for all $\lambda > 0$ and all $i \ne j$,
- $Q\alpha = 0$ for all $\alpha \in \mathbb{R}$, where we identify α with $(\alpha, \dots, \alpha)^T \in \mathbb{R}^d$.

We say that a (possibly nonlinear) operator $Q\colon \mathbb{R}^d \to \mathbb{R}^d$ satisfies the *positive maximum principle* if for $u \in \mathbb{R}^d$ and $i = 1, \dots, d$ it holds $(Qu)_i \le 0$ whenever $u_i \ge u_j$ for all $j = 1, \dots, d$.

Given a measurable space $(\Omega, \mathcal{F})$ and a mapping $\mathcal{E}\colon \mathcal{L}^\infty \to \mathbb{R}$, we say that $(\Omega, \mathcal{F}, \mathcal{E})$ is a *convex expectation space* if there exists a set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$ and a family $(\alpha_\mathbb{P})_{\mathbb{P}\in\mathcal{P}} \subset [0,\infty)$ with $\inf_{\mathbb{P}\in\mathcal{P}} \alpha_\mathbb{P} = 0$ such that

$$\mathcal{E}(X) = \sup_{\mathbb{P}\in\mathcal{P}} \left(\mathbb{E}_\mathbb{P}(X) - \alpha_\mathbb{P}\right)$$

for all $X \in \mathcal{L}^\infty$, where $\mathbb{E}_\mathbb{P}$ denotes the expectation w.r.t. a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ and $\mathcal{L}^\infty$ denotes the space of all bounded random variables $\Omega \to \mathbb{R}$. If $\alpha_\mathbb{P} = 0$ for all $\mathbb{P} \in \mathcal{P}$, we say that $(\Omega, \mathcal{F}, \mathcal{E})$ is a *sublinear expectation space*.

Definition 2. A *convex Markov chain* is a quadruple $(\Omega, \mathcal{F}, \mathcal{E}, (X_t)_{t\ge0})$, where

- $(\Omega, \mathcal{F})$ is a measurable space.
- $X_t\colon \Omega \to \{1, \dots, d\}$ is $\mathcal{F}$-measurable f.a. $t \ge 0$.
- $\mathcal{E} = (\mathcal{E}_1, \dots, \mathcal{E}_d)^T$, where $(\Omega, \mathcal{F}, \mathcal{E}_i)$ is a convex expectation space for all $i = 1, \dots, d$ and $\mathcal{E}(u_0(X_0)) = u_0$. Here and in the following, we make use of the notation
  $$\mathcal{E}(Y) := (\mathcal{E}_1(Y), \dots, \mathcal{E}_d(Y))^T \in \mathbb{R}^d.$$
- $(X_t)_{t\ge0}$ satisfies a nonlinear version of the Markov property under $\mathcal{E}$.

We say that the Markov chain $(\Omega, \mathcal{F}, \mathcal{E}, (X_t)_{t\ge0})$ is *linear* or *sublinear* if the mapping $\mathcal{E}\colon \mathcal{L}^\infty \to \mathbb{R}^d$ is additionally linear or sublinear, respectively.

Main result

Theorem 1. *Let $Q\colon \mathbb{R}^d \to \mathbb{R}^d$ be a mapping. Then the following statements are equivalent:*

a) Q is a convex Q-operator.

b) Q is convex, satisfies the positive maximum principle and $Q\alpha = 0$ for all $\alpha \in \mathbb{R}$, where $\alpha := (\alpha, \dots, \alpha)^T \in \mathbb{R}^d$.

c) There exists a set $\mathcal{P} \subset \mathbb{R}^{d\times d}$ of Q-matrices and a family $f = (f_q)_{q\in\mathcal{P}} \subset \mathbb{R}^d$ of vectors with $\sup_{q\in\mathcal{P}} f_q = f_{q_0} = 0$ for some $q_0 \in \mathcal{P}$ such that

$$Qu_0 = \sup_{q\in\mathcal{P}} (qu_0 + f_q) \tag{1}$$

for all $u_0 \in \mathbb{R}^d$, where the suprema are to be understood componentwise.

d) There exists a convex Markov chain $(\Omega, \mathcal{F}, \mathcal{E}, (X_t)_{t\ge0})$ such that

$$Qu_0 = \lim_{h\searrow0} \frac{\mathcal{E}(u_0(X_h)) - u_0}{h}$$

for all $u_0 \in \mathbb{R}^d$.

In this case, for each $u_0 \in \mathbb{R}^d$, the function $u\colon [0,\infty) \to \mathbb{R}^d$, $t \mapsto \mathcal{E}(u_0(X_t))$ is the unique classical solution $u \in C^1([0,\infty); \mathbb{R}^d)$ to the IVP

$$u'(t) = Qu(t) = \sup_{q\in\mathcal{P}} (qu(t) + f_q), \quad t \ge 0, \tag{2}$$
$$u(0) = u_0.$$

Sketch of the proof.
$d) \Rightarrow b)$: By contradiction (without details).
$b) \Rightarrow a)$: Follows by direct computation.
$a) \Rightarrow c)$: Convex duality.
$c) \Rightarrow d)$: For $q \in \mathcal{P}$ we consider the semigroup S_q to the ODE $u'(t) = qu(t) + f_q$. Then, for fixed $t \ge 0$ we use a partition of the time interval $[0,t]$, which becomes finer and finer, and optimize after each time step. That is, we consider

$$J_n u_0 := \sup_{q\in\mathcal{P}} S_q(2^{-n}t)u_0$$

for $n \in \mathbb{N}_0$ and define

$$(\mathscr{S}(t)u_0)(x) := \lim_{n\to\infty} (J_n^{2^n} u_0)(x).$$

This is the unique solution to the nonlinear ODE (2). An application of a convex version of Kolmogorov's extension theorem (cf. Denk, Kupper, and Nendel (2018)) completes the proof. □

Remarks and Conclusions

- The same equivalence as in Theorem 1 holds in the sublinear case.
- The proof of Theorem 1 shows that every convex Markov semigroup on $\mathbb{R}^d$ is a Nisio semigroup. Moreover, the construction of the semigroup is independent of the dual representation of the operator Q.
- Although Q has an unbounded convex conjugate, the convex initial value problem
  $$u'(t) = Qu(t) \ \text{ for all } t \ge 0, \quad u(0) = u_0. \tag{3}$$
  has a unique global solution.
- Solutions to (3) remain bounded. Therefore, a Picard iteration or Runge-Kutta methods can be used for numerical comutations and the convergence rate (depending on the size of the initial value u_0) can be explicitly computed.
- As in the linear case, by solving the differential equation (3) one can compute expressions of the form
  $$u(t) = \mathcal{E}(u_0(X_t)).$$
  under model uncertainty.

References

Denk, R., M. Kupper, and M. Nendel (2017): "A semigroup approach to nonlinear Lévy processes," *Preprint.*

——— (2018): "Kolmogorov-type and general extension results for nonlinear expectations," *Banach J. Math. Anal.*, 12(3), 515–540.

Nisio, M. (1976/77): "On a non-linear semi-group attached to stochastic optimal control," *Publ. Res. Inst. Math. Sci.*, 12(2), 513–537.