In Search of a Global Belief Model for Discrete-time Uncertain Processes

Natan T'Joens, Jasper De Bock, Gert de Cooman

FLIP, Ghent University, Belgium

GHENT UNIVERSITY

Framework

We consider a *discrete-time uncertain process*, being a sequence $X_1, X_2, \dots, X_n, \dots$ of uncertain states, where the state X_k at each discrete time $k \in \mathbb{N}$ takes values in a fixed non-empty *finite set* $\mathscr{X}$, called the *state space*. For each *situation* $x_{1:n}$, being a finite string $(x_1, \dots, x_n) \in \mathscr{X}_{1:n} := \mathscr{X}^n$ of possible state values, we are given a **local belief model** $\overline{Q}_{x_{1:n}}$ in the form of a *coherent upper expectation* on the set $\mathscr{L}(\mathscr{X})$ of all real-valued (bounded) functions on $\mathscr{X}$.

Local Models $\overline{Q}_{x_{1:n}}$

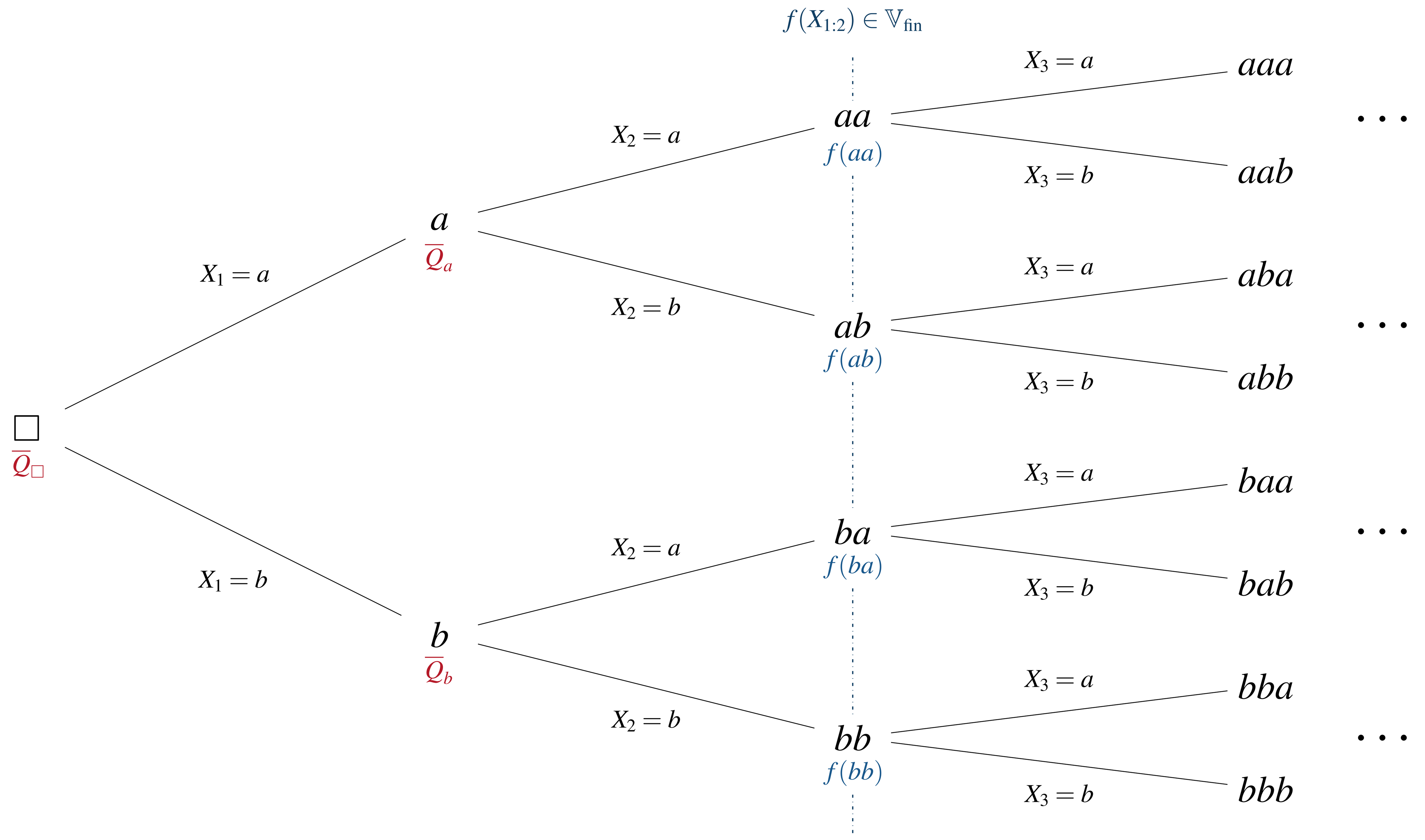
The local models $\overline{Q}_{x_{1:n}}$ express beliefs about how the process changes from one time instant to the next.

Finitary Gambles

Finitary gambles are real-valued functions that depend on the state only at a finite number of time instances. We gather all of them in $\mathbb{V}_{\text{fin}}$.

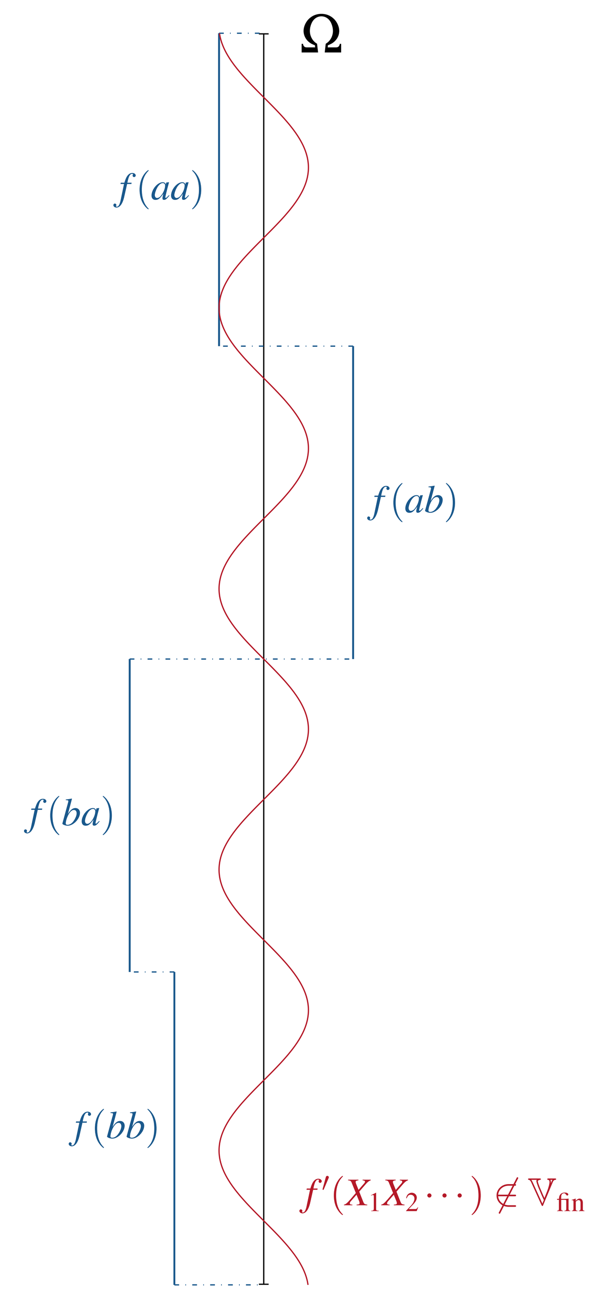
The Sample Space

An infinite sequence $x_1x_2x_3\cdots$ of state values is called a *path* and is denoted by $\omega = x_1x_2x_3\cdots$. The *sample space* $\Omega := \mathscr{X}^{\mathbb{N}}$ is the set of all such paths ω. Moreover, we use $\overline{\mathbb{V}}$ to denote the *set of all global variables, being extended real-valued functions on* Ω.



Global Model $\overline{\mathrm{E}}$?

"How and to which extent, can we extend the information captured in the local models towards global information about the entire process?".



The Axioms

P1.	$\overline{\mathrm{E}}(f(X_{n+1})\vert x_{1:n}) = \overline{Q}_{x_{1:n}}(f)$	for any $f \in \mathscr{L}(\mathscr{X})$ and any $x_{1:n} \in \mathscr{X}^*$.	**[Compatibility with local models]**
P2.	$\overline{\mathrm{E}}(f\vert s) = \overline{\mathrm{E}}(f\mathbb{I}_s\vert s)$	for any $f \in \mathbb{V}_{\text{fin}}$ and any $s \in \mathscr{X}^*$.	**[Conditioning]**
P3.	$\overline{\mathrm{E}}(f\vert X_{1:k}) \leq \overline{\mathrm{E}}(\overline{\mathrm{E}}(f\vert X_{1:k+1})\vert X_{1:k})$	for any $f \in \mathbb{V}_{\text{fin}}$ and any $k \in \mathbb{N}_0$.	**[Weak version of the law of iterated expectations]**
P4.	$f \leq g \Rightarrow \overline{\mathrm{E}}(f\vert s) \leq \overline{\mathrm{E}}(g\vert s)$	for any $f, g \in \overline{\mathbb{V}}$ and any $s \in \mathscr{X}^*$.	**[Monotonicity]**

P5. $\lim_{n\to+\infty} f_n = f \Rightarrow \limsup_{n\to+\infty} \overline{\mathrm{E}}(f_n\vert s) \geq \overline{\mathrm{E}}(f\vert s)$.

Axiom of continuity. For any $f = \lim_{n\to+\infty} f_n$, where $\{f_n\}_{n\in\mathbb{N}_0}$ is a sequence of *finitary gambles* that is uniformly bounded below, we regard $\overline{\mathrm{E}}(f\vert s)$ as an abstract idealisation of $\overline{\mathrm{E}}(f_n\vert s)$ for large n. However, since $\{\overline{\mathrm{E}}(f_n\vert s)\}_{n\in\mathbb{N}_0}$ may not converge, we impose that $\overline{\mathrm{E}}(f\vert s)$ should therefore definitely not exceed the limit superior $\limsup_{n\to+\infty} \overline{\mathrm{E}}(f_n\vert s)$, as this would result in an unwarranted loss of information. This axiom is rather weak since it only imposes an *inequality* and only applies to sequences of *finitary gambles*.

Most Conservative Model

What?

The axioms P1–P5 are *internally consistent*, meaning that, if the local models $\overline{Q}_{x_{1:n}}$ are coherent, there is always at least one global upper expectation $\overline{\mathrm{E}}$ that satisfies P1–P5. Furthermore, it turns out there is moreover a *unique largest* global upper expectation $\overline{\mathrm{E}}^*$ satisfying P1–P5. If we agree that larger upper expectations are more conservative or less informative—this can be justified from both an interpretation in terms of upper envelopes of linear expectations or from a behavioural interpretation in terms of infimum selling prices—then *this model* $\overline{\mathrm{E}}^*$ *is the most conservative belief model that satisfies* P1–P5.

Why?

If one agrees that P1–P5 is desirable for a global upper expectation, the best thing to do, we think, is to choose the most conservative model $\overline{\mathrm{E}}^*$ among those that satisfy P1–P5. Indeed, any other model is necessarily less conservative and therefore adds information that is not implied by our axioms. On the other hand, even if one prefers to impose additional axioms, then still, $\overline{\mathrm{E}}^*$ would serve as a *conservative upper bound* for his desired upper expectation.

Alternative Models

Measure-theoretic models

Instead of using coherent upper expectations to model the local dynamics of a process, we could alternatively describe our beliefs using (closed and convex) sets of probability charges on $\mathscr{X}$. In that way, we could consider the set of all **σ-additive** probability measures on Ω that are compatible with these local models. Alternatively, by extending the local probability charges without the assumption of σ-additivity, we could also consider the set of all **finitely additive** probability measures on Ω that are compatible with the local models. For both cases, the global upper expectation can be defined as the supremum of the expectations corresponding to the compatible σ-additive or finitely additive probability measures on Ω.

Martingale-based model

Global upper expectations can alternatively be defined using the game-theoretic framework of Shafer and Vovk, which relies on the concept of a **supermartingale**: a capital process—the evolution of a subject's capital—that is obtained by betting against a system. Concretely, a supermartingale $\mathscr{M}$ is a map $\mathscr{M}\colon \mathscr{X}^* \to \mathbb{R}$ that is uniformly bounded below and that satisfies $\overline{Q}_s(\mathscr{M}(s\,\cdot)) \leq \mathscr{M}(s)$ **for all** $s \in \mathscr{X}^*$. The global game-theoretic upper expectation operator $\overline{\mathrm{E}}_{\mathrm{V}}\colon \overline{\mathbb{V}} \times \mathscr{X}^* \to \mathbb{R}$ is then defined by

$$\overline{\mathrm{E}}_{\mathrm{V}}(f\vert s) := \inf\{\mathscr{M}(s)\colon (\forall \omega \in \Gamma(s)) \liminf \mathscr{M} \geq f\} \quad \forall f \in \overline{\mathbb{V}} \text{ and } \forall s \in \mathscr{X}^*,$$

where $\mathscr{M}$ is a supermartingale.

Connection

The most conservative model $\overline{\mathrm{E}}^*$ under P1–P5 = **The game-theoretic upper expectation $\overline{\mathrm{E}}_{\mathrm{V}}$.**

Alternative characterisation

Since our operator $\overline{\mathrm{E}}^*$ coincides with the game-theoretic upper expectation $\overline{\mathrm{E}}_{\mathrm{V}}$, we have an *alternative characterisation—and interpretation—for* $\overline{\mathrm{E}}_{\mathrm{V}}$ based on a limited set of intuitive axioms: P1–P5.

Properties

Despite the simplicity of our axioms P1–P5, the global upper expectation $\overline{\mathrm{E}}^*$ turns out to have remarkably *strong mathematical properties*. Most of them are a direct consequence of the fact that it coincides with $\overline{\mathrm{E}}_{\mathrm{V}}$. To name a few: it satisfies a weak and a strong law of large numbers, continuity with respect to non-decreasing sequences, continuity with respect to non-increasing sequences of finitary gambles and a version of Lévy's zero-one law.