Two-State Imprecise Markov Chains for Statistical Modelling of Two-State Non-Markovian Processes

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1. Aims

We aim to present a new framework for

- fitting an imprecise two-state Markov chain
- to data generated by a two-state non-Markovian process
- via an imprecise version of MCMC

2. Old approach

1. transition times $T_i \sim \mathrm{Exp}(\lambda_i)$ (i.e. precise Markov chain assumed)
2. set of priors for λ_i
3. combine with data to get set of posteriors for λ_i
4. posterior predictive bounds on λ_i
5. use these bounds to fix an imprecise Markov chain

3. What is wrong with the old approach?

- Model = set of distributions on parameters of a precise Markov chain. Imprecise Markov chains are not equivalent to this model.
- Sampling uncertainty ignored: only posterior predictive bounds are used. No full uncertainty quantification.
- Imprecision does not reflect violations of stationarity & Markovianity
- Non-Bayesian analysis: inferences are not coherent with the model

4. New approach

1. transition times $T_i \sim f_i(\lambda_i)$ where f_i is a non-Markovian and/or non-stationary process and λ_i are unknown parameters of this process
2. set of prior distributions on λ_i
3. for each prior, use standard MCMC to sample posterior realizations of λ_i i.e. fit the non-Markovian process to the data
4. for each posterior sample of λ_i bound process by an imprecise Markov chain
5. produce sample of posterior bounds for any predictive quantity

5. Advantages to the new approach

- a form of imprecise MCMC: imprecision built into the predictive part this works because of Walley's marginal extension theorem
- fully coherent robust Bayesian approach: predictions are directly derived from posterior distribution
- imprecision reflects lack of data & non-Markovianity & non-stationarity (even if there is lot's of data, inferences can remain imprecise)

6. How did we do it?

Step 1

Model non-Markovian two-state process by a many-state Markov chain.
For example (non-exponential transition from state 2 to state 1):

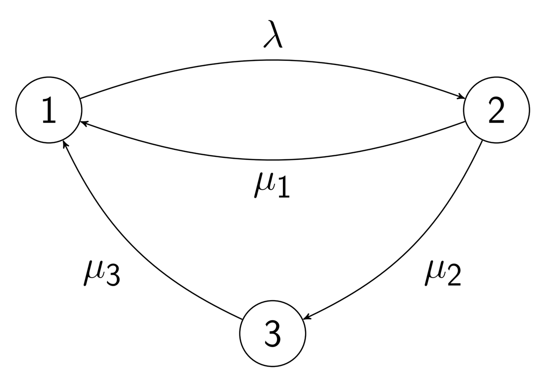


Figure 1: Example of a Markov chain.

with the following transition rate matrix:

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu_1 & -\mu_1 - \mu_2 & \mu_2 \\ \mu_3 & 0 & -\mu_3 \end{bmatrix} \quad (1)$$

Step 2

Fit this many-state Markov chain to the data through its associated phase-type distribution.
How? Use time series data to estimate the parameters λ, μ_1, μ_2 and μ_3.

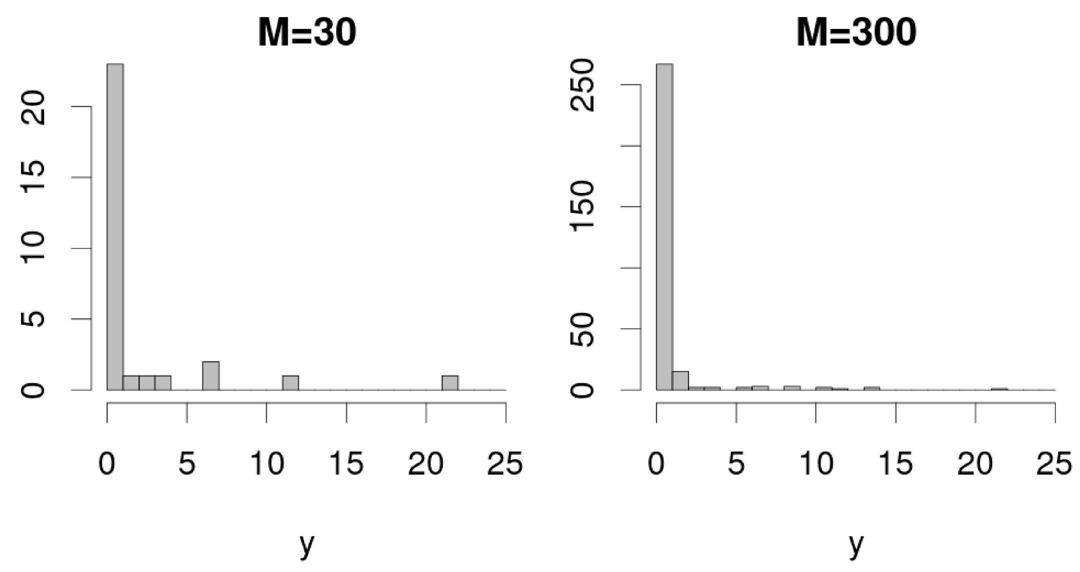


Figure 2: Histogram of times to transition data from state 2 to state 1. The left figure depicts just the first 30 observations, whilst the right figure depicts all 300 observations.

- All parameters fitted using MCMC.
- The model is programmed in Stan.
- Stan conveniently allows direct programming of the phase-type distribution, as well as calculating the limiting lower and upper probabilities for being in state the working state (see step 4).

Step 3

Lump the process to a two-state imprecise Markov chain. Why? Easier to work with.

- We leave state 1 as it is → denoted by α.
- We lump states 2 and 3 together → denoted by β.

$$[\underline{Q}f]_\alpha = \min\{\lambda_*(f_\beta - f_\alpha), \lambda^*(f_\beta - f_\alpha)\}, \quad (2)$$
$$[\underline{Q}f]_\beta = \min\{\mu_*(f_\alpha - f_\beta), \mu^*(f_\alpha - f_\beta)\}. \quad (3)$$

Then, with the λ and μ that achieve the minimum in eqs. (2) and (3):

$$(\lambda_f, \mu_f) \begin{cases} (\lambda_*, \mu^*) & \text{if } f_\alpha \leq f_\beta \\ (\lambda^*, \mu_*) & \text{if } f_\alpha > f_\beta \end{cases} \quad (4)$$

and

$$Q_f := \begin{bmatrix} -\lambda_f & \lambda_f \\ \mu_f & -\mu_f \end{bmatrix} \quad (5)$$

we see that we can interpret our lumped process as one with a precise transition rate

$$\lambda_* = \lambda^* = -Q_{11} \quad (6)$$

for going from state 1 to state 2, and lower and upper transition rates

$$\mu_* = \min_{i=2}^{n} Q_{i1} = \min\{\mu_1, \mu_3\} \qquad \mu^* = \max_{i=2}^{n} Q_{i1} = \max\{\mu_1, \mu_3\} \quad (7)$$

for going from state 2 to state 1.

Step 4

The limiting lower and upper probabilities $(\underline{\pi}, \overline{\pi})$ for being in state 1 i.e. the working state, which are typical inferential quantities of interest:

$$\underline{\pi} := \lim_{t\to\infty} \underline{T}_t \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\mu_*}{\lambda + \mu_*} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (8)$$

$$\overline{\pi} := \lim_{t\to\infty} \overline{T}_t \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\mu^*}{\lambda + \mu^*} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (9)$$

λ, μ_1, μ_2 and μ_3 are now treated as uncertain quantities, so are $\underline{\pi}$ and $\overline{\pi}$. In particular, given a prior distribution for each of the parameters:

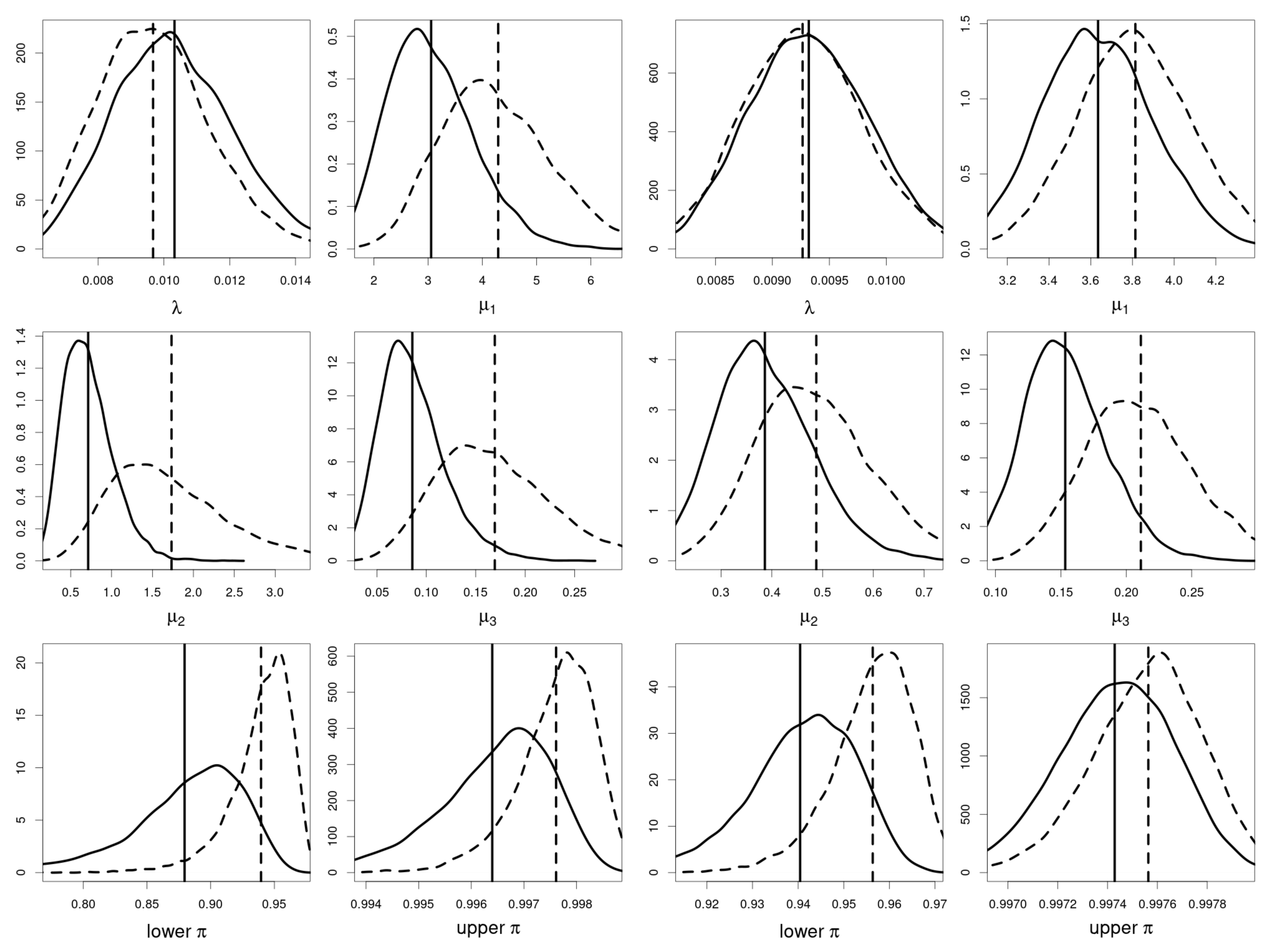
$$\lambda \sim \mathrm{Gamma}(s, s\tau_0) \quad (10)$$
$$\mu_i \sim \mathrm{Gamma}(s, s\tau_i) \qquad i \in \{1, 2, 3\} \quad (11)$$

Aim: Derive a set of posterior distributions for $\underline{\pi}$ and $\overline{\pi}$.
For a specific reliability application, the relevant extreme cases are the ones obtained for

$$(\tau_0, \tau_1, \tau_2, \tau_3) \in \{(50, 1, 4, 40), (250, 0.1, 0.2, 2)\} \quad (12)$$

We simply plot the results under these two priors, as these cover the two relevant extreme cases.

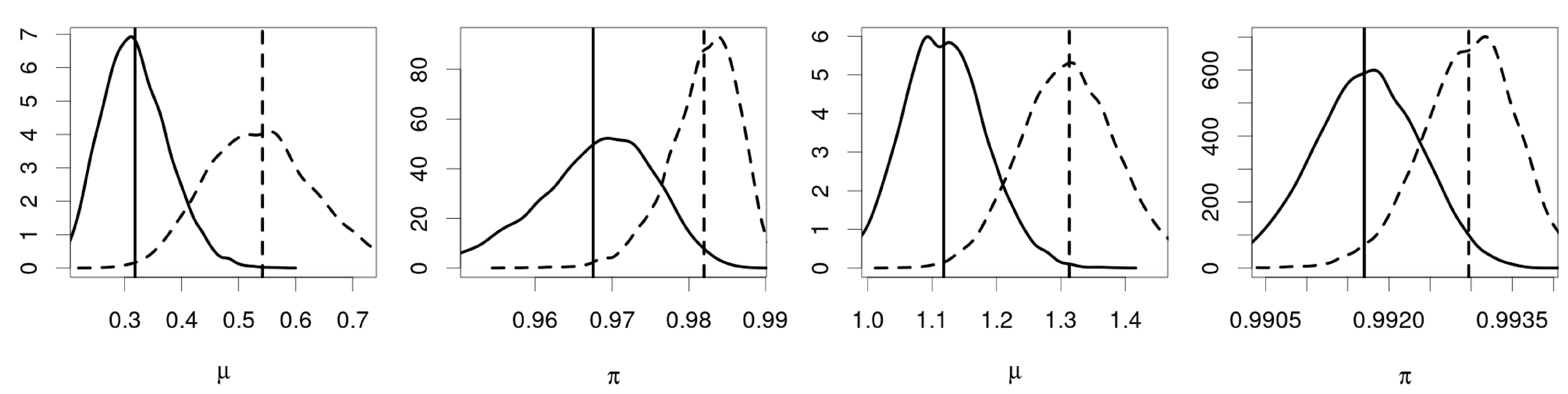
7. Results



(a) For small sample size ($N = M = 30$). (b) For large sample size ($N = M = 300$).

Figure 3: Posterior densities for the lumped model under both priors (solid and dashed line). The vertical lines indicate posterior expectations.

8. Comparison With Two-State Exponential Model



(a) For small sample size ($N = M = 30$). (b) For large sample size ($N = M = 300$).

Figure 4: Posterior densities for the exponential model under both priors (solid and dashed line). The vertical lines indicate posterior expectations.

9. Conclusions

- We have improved the way imprecise Markov chains are fitted
- Imprecision can result not only from limited data but also from characteristics of the process
- These principles may apply to fitting of general stochastic processes (not only imprecise Markov chains)
- We identified a class of problems where imprecise MCMC is easy
- Larger problems may require imprecise ABC if likelihood of phase-type distribution has no closed form

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