Interpreting, Axiomatising and Representing Coherent Choice Functions in Terms of Desirability

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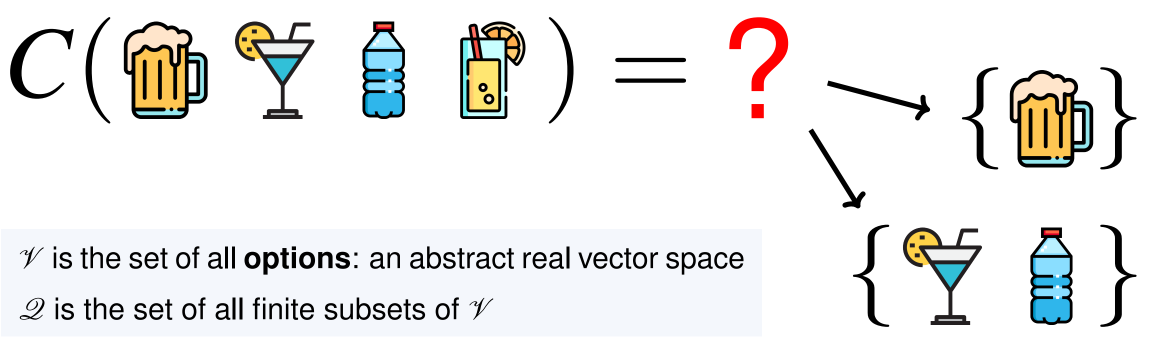

a unifying framework for modelling set-valued choice!

$\mathscr{V}$ is the set of all **options**: an abstract real vector space

$\mathscr{Q}$ is the set of all finite subsets of $\mathscr{V}$

a **choice function** C is a map from $\mathscr{Q}$ to $\mathscr{Q}$ such that $C(A) \subseteq A$ for every $A \in \mathscr{Q}$

the corresponding **rejection function** R is defined by $R(A) := A \setminus C(A)$, for all $A \in \mathscr{Q}$

desirability provides an interpretation for each of our models

	D	K	R / C
	an option $v \in \mathscr{V}$ is desirable if it is strictly preferred to zero	an option set $A \in \mathscr{Q}$ is desirable if it is thought to contain at least one desirable option	an option $u \in A$ is rejected from A if at least one option $v \in A$ is strictly preferred over u, in the sense that $v - u$ is desirable
	Set of desirable options	Set of desirable options sets	Rejection function / Choice function
COHERENT	D_1 $0 \notin D$ D_2 $\mathscr{V}_{\succ 0} \subseteq D$ ① ② D_3 if $u, v \in D$ and $(\lambda, \mu) > 0$, then $\lambda u + \mu v \in D$	K_0 if $A \in K$ then also $A \setminus \{0\} \in K$, for all $A \in \mathscr{Q}$ K_1 $\{0\} \notin K$ K_2 $\{u\} \in K$, for all $u \in \mathscr{V}_{\succ 0}$ K_3 if $A_1, A_2 \in K$ and if, for all $u \in A_1$ and $v \in A_2$ $(\lambda_{u,v}, \mu_{u,v}) > 0$, then also $\{\lambda_{u,v} u + \mu_{u,v} v \colon u \in A_1, v \in A_2\} \in K$ K_4 if $A_1 \in K$ and $A_1 \subseteq A_2 \in \mathscr{Q}$, then also $A_2 \in K$	R_0 for all $A \in \mathscr{Q}$ and $u \in A$: $u \in R(A) \Leftrightarrow 0 \in R(A - u)$ R_1 $R(\emptyset) = \emptyset$, and $R(A) \neq A$ for all $A \in \mathscr{Q} \setminus \{\emptyset\}$ R_2 $0 \in R(\{0, u\})$, for all $u \in \mathscr{V}_{\succ 0}$ R_3 if $A_1, A_2 \in \mathscr{Q}$, $0 \in R(A_1 \cup \{0\})$ and $0 \in R(A_2 \cup \{0\})$ and if $(\lambda_{u,v}, \mu_{u,v}) > 0$ for all $u \in A_1$ and $v \in A_2$, then $0 \in R(\{\lambda_{u,v} u + \mu_{u,v} v \colon u \in A_1, v \in A_2\} \cup \{0\})$ R_4 if $A_1 \subseteq A_2$ then $R(A_1) \subseteq R(A_2)$, for all $A_1, A_2 \in \mathscr{Q}$
TOTAL	$\mathrm{D_T}$ for all $u \in \mathscr{V} \setminus \{0\}$, either $u \in D$ or $-u \in D$	$\mathrm{K_T}$ $\{u, -u\} \in K$ for all $u \in \mathscr{V} \setminus \{0\}$	$\mathrm{R_T}$ $0 \in R(\{0, u, -u\})$, for all $u \in \mathscr{V} \setminus \{0\}$
MIXING	$\mathrm{D_M}$ if $\mathrm{posi}(A) \cap D \neq \emptyset$, then also $A \cap D \neq \emptyset$, for all $A \in \mathscr{Q}$ ③	$\mathrm{K_M}$ if $B \in K$ and $A \subseteq B \subseteq \mathrm{posi}(A)$, then also $A \in K$, for all $A, B \in \mathscr{Q}$	$\mathrm{R_M}$ if $A \subseteq B \subseteq \mathrm{conv}(A)$ then also $R(B) \cap A \subseteq R(A)$, for all $A, B \in \mathscr{Q}$ ④
ARCHIMEDEAN ⑤	$\mathrm{D_A}$ for all $u \in D$, there is an $\varepsilon \in \mathbb{R}_{>0}$ such that $u - \varepsilon \in D$	$\mathrm{K_A}$ for all $A \in K$, there is an $\varepsilon \in \mathbb{R}_{>0}$ such that $A - \varepsilon \in K$	$\mathrm{R_A}$ for all $A \in \mathscr{Q}$ and $u \in \mathscr{V}$ such that $u \in R(A \cup \{u\})$, there is some $\varepsilon \in \mathbb{R}_{>0}$ such that $u \in R((A - \varepsilon) \cup \{u\})$

OTHER PROPERTIES?

$K_D := \{A \in \mathscr{Q} \colon A \cap D \neq \emptyset\}$

$D_K := \{u \in \mathscr{V} \colon \{u\} \in K\}$

loss of information! (unless binary)

$\{v - u \colon v \in A\}$ ‖ $u \in R(A \cup \{u\}) \Leftrightarrow A - u \in K$

an **intersection** of sets of desirable option sets K amounts to taking the **union** of the corresponding choice functions C

K is ☆ $\Leftrightarrow$ $K = \cap\{K_D \colon D \in \mathscr{D}\}$ ⑥

for some non-empty set $\mathscr{D}$ of ☆ sets of desirable options

	COHERENT	COHERENT TOTAL	COHERENT MIXING	COHERENT ARCHIMEDEAN	COHERENT MIXING ARCHIMEDEAN
$C \Leftrightarrow$	set of strict (partial) preference orders	set of strict total preference orders	set of lexicographic preference orders	closed set of lower previsions / lower expectations / credal sets	closed set of linear previsions / probability measures

① $\mathscr{V}_{\succ 0}$ is a convex cone in $\mathscr{V} \setminus \{0\}$ whose elements must be desirable

② $\lambda \geq 0$, $\mu \geq 0$ and $\lambda + \mu > 0$

③ $\mathrm{posi}(A) := \left\{\sum_{k=1}^{n} \lambda_k u_k \colon n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, u_k \in A\right\}$

④ $\mathrm{conv}(A) := \left\{\sum_{k=1}^{n} \lambda_k u_k \colon n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, \sum_{k=1}^{n} \lambda_k = 1, u_k \in A\right\}$

⑤ (so far) only for $\mathscr{V} = \mathscr{L}(\mathscr{X})$: the set of all bounded real-valued functions (gambles) on some set $\mathscr{X}$; $\mathscr{V}_{\succ 0} = \{u \in \mathscr{L}(\mathscr{X}) \colon \inf u > 0\}$

⑥ $\mathscr{D}$ is closed in the ARCHIMEDEAN cases; how should we modify archimedeanity for $\mathscr{D}$ to not be closed?

Choice models: from linear option spaces to sets of horse lotteries

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Choosing between options

We consider a subject's choices between options in a set, modelled by a rejection function:

$R(\{🍺, 🍸, 🧴, 🥤\}) = \{🧴, 🥤\} \longrightarrow$ rejected options from the set $\{🍺, 🍸, 🧴, 🥤\}$

Essential aspects:
- incomparability: more than one option may remain unrejected.
- binary choice is only a special case: $R(\{🧴, 🥤\}) = \{🥤\}$.

Modelling uncertainty

To model a subject's uncertainty about the unknown value of some variable X in a set $\mathscr{X}$:

let her choose between rewards that depend on the value of X.

The subject now chooses between uncertain rewards $o(X)$:

option $o \colon \mathscr{X} \to \mathscr{R}$ (finite set of rewards $\mathscr{R}$)

$R(\{o_1(X), \dots, o_n(X)\}) = ?$

Horse lotteries as options

To ensure that there are enough options for the subject to choose between, we randomise them, so take all their convex mixtures:

let her choose between lotteries that depend on the value of X.

The subject now chooses between horse lotteries $H(X)$:

horse lottery $H \colon \mathscr{X} \to \Delta(\mathscr{R})$ (set of all mass functions on $\mathscr{R}$)

$R(\{H_1(X), \dots, H_n(X)\}) = ?$

Options in a linear space

Recent work by Van Camp (2017) en De Bock and De Cooman (2018, 2019) has led to general representation and conservative inference theorems for choice on abstract ordered linear spaces of options.

Why work with general ordered linear option spaces?
- mathematical convenience: linearity less cumbersome than convexity
- modelling indifference: options are affine subspaces
- modelling infinite exchangeability: spaces of Bernstein polynomials
- non-standard orderings: quantum mechanics, ...

How to connect the two?

Abstract options
- an ordered linear space of options $\mathscr{V}$
- the set of all finite option sets $\mathscr{Q}(\mathscr{V})$
- a rejection function $R \colon \mathscr{Q}(\mathscr{V}) \to \mathscr{Q}(\mathscr{V}) \colon A \mapsto R(A)$

The corresponding set of desirable option sets

$K := \{A - u \colon u \in R(A \cup \{u\}), u \in \mathscr{V}, A \in \mathscr{Q}(\mathscr{V})\}$

is the set of all option sets B such that 0 is rejected from $B \cup \{0\}$.

De Bock and De Cooman (2018, 2019) have proved many representation and inference results for such K that are coherent, mixing, total, or Archimedean.

Connection

Consider the linear option space

$\mathscr{D} := \mathrm{span}(\mathscr{H} - \mathscr{H}) = \{\lambda(H - G) \colon \lambda > 0, H, G \in \mathscr{H}\}$.

When we begin with an R^* on $\mathscr{H}$, we let, up to scaling:

$K_{R^*} :=^* \{A^* - H \colon A^* \in \mathscr{Q}^*, H \in \mathscr{H}, H \in R^*(A^* \cup \{H\})\}$.

When we begin with a K on $\mathscr{D}$, we let:

$R^*_K(\emptyset) := \emptyset$ and $H \in R^*_K(A^* \cup \{H\})$ when $A^* - H \in K$, for all $A^* \in \mathscr{Q}^*$ and $H \in \mathscr{H}$.

Horse lotteries
- the set of all horse lotteries $\mathscr{H}$
- the set of all finite option sets $\mathscr{Q}^*$

A rejection function $R^* \colon \mathscr{Q}^* \to \mathscr{Q}^* \colon A^* \mapsto R^*(A^*)$ on horse lotteries is called total if it is coherent and also satisfies

R^*_T. $\frac{H_1 + H_2}{2} \in R^*(\{H_1, \frac{H_1 + H_2}{2}, H_2\})$ for all $H_1, H_2 \in \mathscr{H}$ such that $H_1 \neq H_2$.

It is called mixing if it is coherent and also satisfies

R^*_M. if $A^* \subseteq B^* \subseteq \mathrm{conv}(A^*)$ then $R(B^*) \cap A^* \subseteq R(A^*)$, for all $A^*, B^* \in \mathscr{Q}^*$.

Connection theorems

Theorem (Isomorphism). *Consider any coherent rejection function R^* on $\mathscr{H}$ and any coherent set of desirable option sets K on $\mathscr{D}$. Then $K = K_{R^*} \Leftrightarrow R^* = R^*_K$.*

Theorem (Preservation of properties). *Let K be any set of desirable option sets on $\mathscr{D}$, and let R^* be any rejection function on $\mathscr{H}$.*

(i) *if K is coherent, then so is R^*_K; and if R^* is coherent, then so is K_{R^*};*

(ii) *if K is total, then so is R^*_K; and if R^* is total, then so is K_{R^*};*

(iii) *if K is mixing, then so is R^*_K; and if R^* is mixing, then so is K_{R^*}.*

Theorem (Preservation of infima). (i) *Let K_i, $i \in I$ be an arbitrary non-empty family of sets of desirable option sets on $\mathscr{D}$, and let $K := \bigcap_{i \in I} K_i$ be its intersection. Then $R^*_K = \bigcap_{i \in I} R^*_{K_i}$.*

(ii) *Let R^*_i, $i \in I$ be an arbitrary non-empty family of coherent rejection functions on $\mathscr{H}$, and let $R^* := \bigcap_{i \in I} R^*_i$ be its infimum. Then $K_{R^*} = \bigcap_{i \in I} K_{R^*_i}$.*

Representation theorems

Theorem (Representation for coherence). *A rejection function on horse lotteries R^* is coherent if and only if it is the intersection of some non-empty collection of coherent binary rejection functions. The largest such collection is the set of all coherent binary rejection functions that dominate it.*

Theorem (M-admissibility). *A rejection function on horse lotteries R^* is total if and only if it is the intersection of some non-empty collection of total binary rejection functions. The largest such collection is the set of all total binary rejection functions that dominate it.*

Theorem (E-admissibility). *A rejection function on horse lotteries R^* is mixing (and Archimedean) if and only if it is the intersection of some non-empty (closed) collection of mixing (and Archimedean) binary rejection functions. The largest such collection is the set of all mixing (and Archimedean) binary rejection functions that dominate it.*

Inference methods

Coherence, totality and mixingness are preserved under taking arbitrary non-empty intersections.

For each of these notions:
- consistency
- inferential closure
- conservative inference (natural extension)

The binary models serve as the (dually) atomic—complete?—ones:
- intersection of comparable binary models leads to binary models
- intersection of incomparable binary models leads to non-binary models

The representation theorems can be seen as soundness and completeness results for (semantics of) these logics, defined by the (implicit) inference rules in the coherence, totality and mixingness definitions.