Bernstein's socks, polynomial-time provable coherence and entanglement

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Who?

The *real* Mad Hatter: senior researcher at CSIS, U. Limerick, but until the other day prof. at IDSIA

The *speaker*: a convenience logician*, currently at IDSIA

*Concept and formulation by Yoichi Hirai

Prof. (and scientific co-director) at IDSIA, who some time ago told the two others “but all this IP stuff *is* logic, don’t you think?”
And everything started.

Message of the paper / poster

- Quantum weirdness, such as the violation of Bell's inequalities or entanglement, is not inherent to Quantum Mechanics as such but to any theory of bounded rationality based on the requirement that checking its coherence should be an easy task, of which QM is a just a particular instance.

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Theory of rationality

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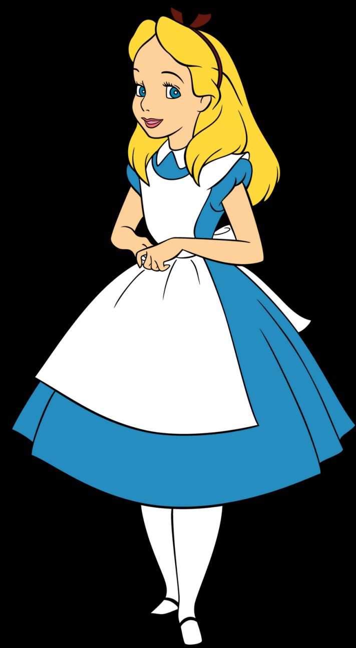
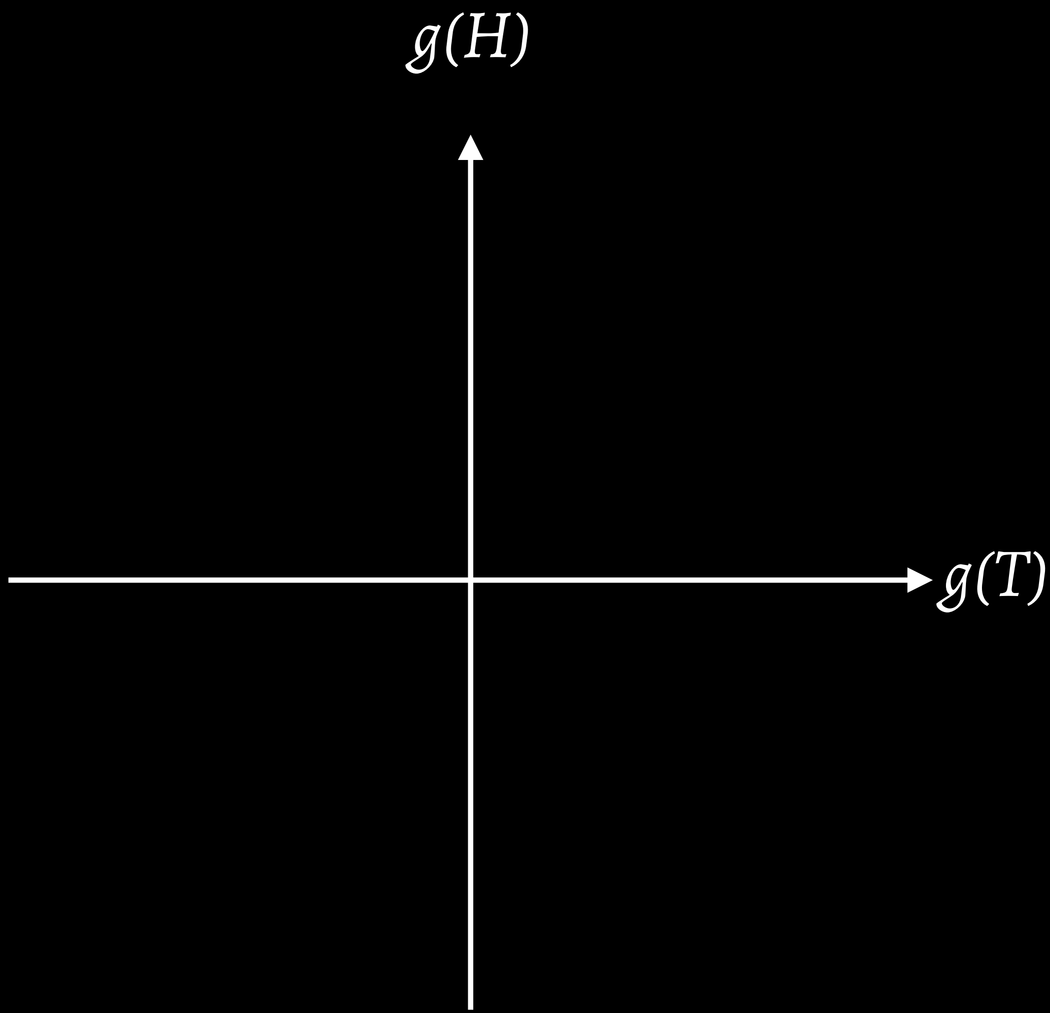
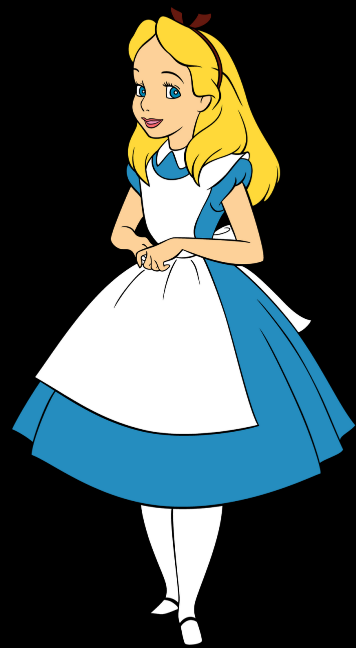


Theory of (almost) desirable gambles





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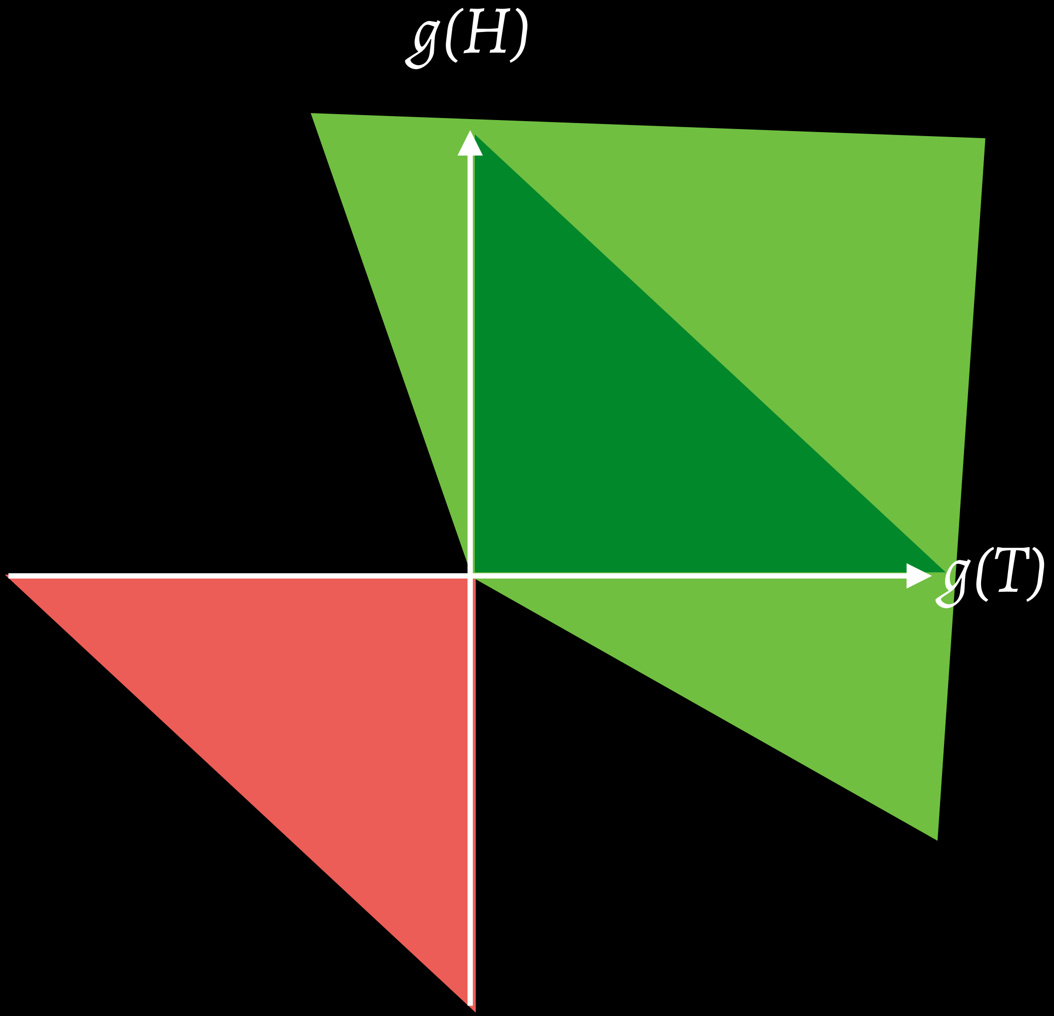

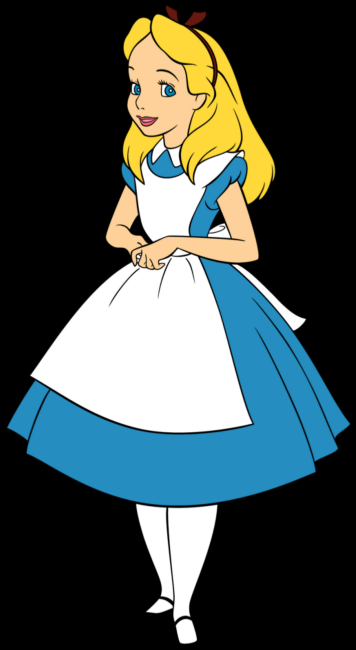

TADG as a logic calculus

Ax.: $$\frac{}{A \triangleright g} \quad g > 0$$ **(accepting partial gain)**

Rule 1: $$\frac{A \triangleright g \qquad A \triangleright f}{A \triangleright \lambda g + \mu f} \quad \lambda, \mu \geq 0$$ **(conical hull)**

Rule 2: $$\frac{\{ A \triangleright g\, \varepsilon^k \mid k > 0 \}}{A \triangleright g} \quad \varepsilon \in (0,1)$$ **(closure)**

Rule 3: $$\frac{A \triangleright -1}{A \triangleright f}$$ **(ex falso)**

(Plus usual structural rules for sequents - reflexivity is enough)

A completeness theorem for TADG

$\nvdash A \triangleright -1$

if and only if

there is some probability *p*:

$p \Vdash g, \ \forall g \in A$

$$p \Vdash g := \int_{\Omega} g(\omega) dp(\omega) \geq 0$$

A is logically consistent / coherent

The consistency (coherence) problem

Consider a finite set of assessments A:

Does $\not\vdash A \triangleright -1$?

Is A coherent / logical consistent ?

When the possibility space is infinite, this problem is either impossible [**undecidable**] or difficult [**NP-hard**]

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$$\text{Does } \nvdash A \triangleright -1 \,?$$

When the possibility space is infinite, this problem is either impossible [**undecidable**] or difficult [**NP-hard**]

A theory of bounded [*computational*] rationality

Ax.: $$\frac{}{A \rhd g}$$ *$g \in \Sigma \subset \mathcal{L}^{>}$ and membership is in P*

Rule 1: $$\frac{A \rhd g \qquad A \rhd f}{A \rhd \lambda g + \mu f}$$ $\lambda, \mu \geq 0$ **(conical hull)**

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A theory of bounded [*computational*] rationality

Ax.: $\dfrac{}{A \triangleright g}$ $g \in \Sigma \subset \mathscr{L}^{>}$ ***and membership is in P***

Rule 1:

For a theory of bounded [*computational*] rationality, the consistency problem is decidable in PTIME

Rule 2: $\dfrac{\quad}{A \triangleright g}$

Rule 3: $\dfrac{A \triangleright -1}{A \triangleright f}$ **(ex falso)**

(Plus usual structural rules for sequents - reflexivity is enough)

A clash of two worlds

Whenever $\Sigma \subsetneq \mathscr{L}^{>}$

$$\nvdash A \rhd -1$$

if and only if

there is some probability p:

$$p \Vdash g, \ \forall g \in A$$

The fundamental theorem of *weirdness*

- Let C be a logically consistent set of gambles in a theory T of computational rationality.
  Then the following are equivalent claims:
  1. C includes a negative gamble g that is not in $-\Sigma$ (i.e. it is not "negative" according to T)
  2. C has no classical probabilistic interpretation (model)
  3. Any unit preserving positive linear functional (state) L on the space of gambles that is a model of C ($L \Vdash g, \ \forall g \in C$) is not (a limit of) a mixture of classical evaluation functionals
  4. Any charge extending a state L that is a model of C is necessarily signed (negative probability)

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QM, Bernstein and the weird

- Quantum mechanics is an example of a theory of computational rationality.
- The fundamental theorem of weirdness holds, and quantum weirdness is a consequence of it

- QM is **not** the only theory of computational rationality
- Therefore there are other contexts in which weird things such as entanglement can occur (at least in a thought experiment).

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QM, Bernstein and the weird

Example in our work

- the space of Bernstein's polynomials,
- "being nonnegative" as satisfying the Krivine-Vasilescu's nonnegativity certificate
- a thought experiment uncovering entanglement with two classical coins ("Bernstein's socks")

rationality

- Therefore there are other contexts in which weird things such as entanglement can occur (at least in a thought experiment).

Bernstein's Socks, Polynomial-Time Coherence and Entanglement

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Take Away Message

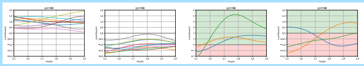
- Weirdness (e.g. Bell's inequality, entanglement) is a consequence of computational rationality, i.e. of imposing to a model of rational choice that its coherence problem is decidable in PTIME

Alice in Classicaland, or (Full) Rationality

- *Possibility space* as a possibly infinite set Ω, its elements as input data for some preparation procedure (for composite system, cartesian product $\Omega_1 \times \ldots \times \Omega_n$)
- *Observables* as bounded real functions - gambles - on $\Omega : g \in \mathscr{L}(\Omega)$.

- **How to enforce rational behaviour in Classicaland?**
- Given an experiment, Alice is a *rational* agent if the set of gambles she accepts
  - A0: contains all nonnegative gambles
  - A1: is a (closed) convex cone
  - A2: does not include a negative function, that is a sure loss

- *Example*: How tall was Albert Einstein? Do you want to bet on it in Classicaland?



(A) (B) (C) (D)

A gamble is a bounded function on Ω=[1.5,2]m.

- If Alice is rational, by A0 she should accept any gamble in Plot A, and by A2 she should not accepts any gamble in Plot B.
- If rational, if Alice accepts **blue** and **orange** gambles in C, then she also accepts **green**.
- In plot D, however she is irrational: since she accepts **blue** and **orange** gambles, by A1 she is forced to accept the **green** one, which is a sure loss and thus violates A2.

- **Rationality in Classicaland as a logic calculus:**
  - A0: $\vdash \emptyset \triangleright g$, for every nonnegative g
  - A1.1: $\{\mathscr{G} \triangleright f, \mathscr{G} \triangleright g\} \vdash \mathscr{G} \triangleright \mu f + \lambda g$, for $\mu, \lambda \geq 0$ // A1.2: $\{\mathscr{G} \triangleright \lambda^k g \mid k>0, \lambda \in (0,1)\} \vdash \mathscr{G} \triangleright g$
  - A2: $\mathscr{G} \triangleright f \vdash \mathscr{G} \triangleright g$, where f is negative and g arbitrary
- **Probabilistic interpretation (sound and complete)**: for every set of gambles $\mathscr{G} \cup \{g\}$

$\vdash \mathscr{G} \triangleright g \iff$ for every finitely additive positive measure μ on Ω, if $L_\mu \Vdash \mathscr{G}$ then $L_\mu \Vdash g$

where $L(g) := \int_\Omega g(\omega) d\mu(\omega)$

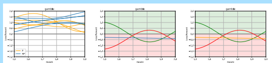
The Coherence Problem

- Let $\mathscr{G} \subseteq \mathscr{L}(\Omega)$ be finite set $\mathscr{G}$ of gambles, and $\mathscr{K}$ its deductive closure
  - $\mathscr{K}$ is not coherent $\iff \exists \lambda_i \geq 0 : -1 - \sum_{\ell=1}^{|\mathscr{G}|} \lambda_\ell g \in \mathscr{L}^{\geq}$
- On $\mathscr{L}(\Omega)$ with Ω infinite, the coherence problem is **not decidable**
- If we restrict the class of gambles to the class of all multivariate polynomials of degrees k bounded, the problem remains in general difficult (**NP hard**)
  - Classical probability theory, when Ω is infinite, is either undecidable or NP-hard

The Mad Hatter in Posiland, or Computational Rationality

- Imagine a word where a gamble is `nonnegative`/`negative` if it is nonnegative/negative and its nonnegativity/negativity can be assessed efficiently (in PTIME)

- **How to enforce rational behaviour in Posiland?**
- Given an experiment, the Mad Hatter is *rational* if the set of gambles he accepts
  - B0: contains all `nonnegative` gambles
  - B1=A1: is a (closed) convex cone
  - B2: does not include a `negative` function, that is sure loss
- *Example reloaded*: How tall was Albert Einstein? Do you want to bet on it in Posiland? Assume we split the nonnegative gambles in two groups [plot A]: the **orange** ones whose nonnegativity can be assessed in polynomial time; and the **blue** ones whose nonnegativity cannot be assessed in polynomial time
  - In Plot B, Alice is P-rational, since the blue gamble does not contradicts B2
  - In Plot C, however, Alice is not P-rational, since she contradicts B2.



(A) (B) (C)

- **Rationality in Posiland as a logic calculus:** (Assume `nonnegative` is closed and contains constant)
  - B0: $\vdash \emptyset \triangleright g$, for every `nonnegative` g
  - B1.1: $\{\mathscr{G} \triangleright f, \mathscr{G} \triangleright g\} \vdash \mathscr{G} \triangleright \mu f + \lambda g$, for $\mu, \lambda \geq 0$ // B1.2: $\{\mathscr{G} \triangleright \lambda^k g \mid k>0, \lambda \in (0,1)\} \vdash \mathscr{G} \triangleright g$
  - B2: $\mathscr{G} \triangleright f \vdash \mathscr{G} \triangleright g$, $\mathscr{G} \triangleright f \vdash \mathscr{G} \triangleright g$, where f is `negative` and g arbitrary

Generalised Expectation and Models

Whenever L is a bounded linear functional on some space of gambles with $L(1)=1$, and $L(g) \geq 0$, we write $L \Vdash g$ (*)

Coherence and Duality

Given a (P) coherent set of gambles $\mathscr{C} \subseteq \mathscr{L}'$, its dual is the class of models (states) satisfying $\mathscr{C}$:

$\{L \in \mathscr{L}'^* \mid L(1) = 1,\ L \Vdash g, \forall g \in \mathscr{C}\}$

Fundamental Theorem of Weirdness

- *Theorem 1*: Let $\mathscr{C}$ be a P-coherent set of gambles. Then the following are equivalent claims:
  1. $\mathscr{C}$ includes a negative gamble g that is not `negative`
  2. $\mathscr{C}$ has no classical probabilistic interpretation (model, as given by (*))
  3. Any positive linear functional L on $\mathscr{L}_R$, preserving the unit ($L(1) = 1$) and satisfying (*) is not a mixture or a limit of mixtures of classical evaluation functionals
  4. Any charge extending a positive linear functional L on $\mathscr{L}_R$, preserving the unit and satisfying (*) is necessarily signed (negative probability)

Introducing "Bernstein's Socks (Coins)"

- Consider two classical coins and the possibility space given by the probabilities of the four possible outcomes H_1H_2, T_1H_2, H_1,T_2, T_1T_2

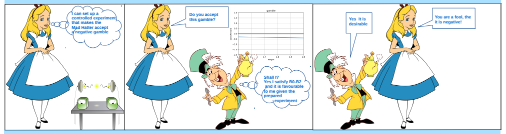
$$\Omega := \{\theta \in \mathbb{R}^3 \mid \theta \geq 0,\ \theta_{H_1H_2} + \theta_{T_1H_2} + \theta_{H_1T_2} \leq 1\}$$

$$\begin{bmatrix} \theta_{H_1H_2} \\ \theta_{T_1H_2} \\ \theta_{H_1T_2} \\ 1 - \theta_{H_1H_2} - \theta_{T_1H_2} - \theta_{H_1T_2} \end{bmatrix} = \text{Prob} \begin{bmatrix} H_1H_2 \\ T_1H_2 \\ H_1T_2 \\ T_1T_2 \end{bmatrix}$$

- The gambles are all real polynomials on Ω of degree 2
- Evaluating the nonnegativity of g is NP-hard. We thus redefine the meaning of being nonnegative as follows: a gamble is said to be `nonnegative` if it is of the form

$$\sum_{\alpha \in \mathbb{N}^4, \sum_{i=1}^4 \alpha_i = 2} u_\alpha \theta_{H_1H_2}^{\alpha_1} \theta_{T_1H_2}^{\alpha_2} \theta_{H_1T_2}^{\alpha_3} (1 - \theta_{H_1H_2} - \theta_{T_1H_2} - \theta_{H_1T_2})^{\alpha_4}, \text{ with } u_\alpha \in \mathbb{R}_{\geq}$$

- `Nonnegativity` can be assessed in PTIME



Bell with Two Classical Coins

- We design a thought experiment in which a negative but not `negative` gamble has positive expectation (a CHSH-like experiment, like in the comic strip, but with classical coins)
- L is a state, i.e. a unit-preserving positive linear functional on Ω
- The set $\mathscr{C} := \{g \in \mathscr{L}_R \mid L \Vdash g\}$ is P-coherent
- Let $1/6 \geq \varepsilon > 0$, and consider $q(\theta) = -(\theta_{H_1H_2} + \theta_{T_1H_2})^2 - (\theta_{H_1H_2} + \theta_{H_1T_2})(-2\theta_{H_1H_2} - 2\theta_{T_1H_2} + 1) - \varepsilon$
  - $L(q) = 1/6 - \varepsilon \geq 0$, meaning that $q \in \mathscr{C}$ and thus q is not `negative`
- q is negative, indeed it holds that $q(\theta) < -\varepsilon$, for every $\theta \in \Omega$
- By Theorem 1, there is no classical probabilistic interpretation (model) for $\mathscr{C}$, hence from the classical point of view, $\mathscr{C}$ is "incoherent"

$$\begin{aligned} L(\theta_{H_1H_2}) &= z_{100} = 1/3 & L(\theta_{H_1H_2}^2) &= z_{200} = 1/3 \\ L(\theta_{T_1H_2}) &= z_{010} = 1/6 & L(\theta_{T_1H_2}^2) &= z_{020} = 0 \\ L(\theta_{H_1T_2}) &= z_{001} = 1/6 & L(\theta_{H_1T_2}^2) &= z_{002} = 0 \\ L(\theta_{H_1H_2}\theta_{T_1H_2}) &= z_{110} = 0 & L(\theta_{H_1H_2}\theta_{H_1T_2}) &= z_{101} = 0 \\ L(\theta_{T_1H_2}\theta_{H_1T_2}) &= z_{011} = 1/6 & L(1) &= z_{000} = 1. \end{aligned}$$

Updating via Partition of Unit

A partition of unit is a family Π of nonnegative functions that sum up to one. Let π a subset sum of elements in Π. Then an updated lower prevision for q is

$$\underline{E}_B(q \mid \pi) = \sup_{\lambda_j \geq 0, \lambda_0} \lambda_0 \ \text{s.t}\ (q - \lambda_0)\pi - \sum_{j=1}^{|G|} \lambda_j g_j$$

Entanglement with Two Classical Coins

- Two coins in the joint state L are drawn from a bag: one is sent to Anne and one to Bob.
- We check that a measurement of the bias of Anne's coin will allow the prediction with certainty of the bias of Bob's coin.
- Assume Alice tosses her coin, and it lands H. Given $\pi=\theta_{HH}+\theta_{HT}$, for gamble $q(\theta)=\theta_{HH}+\theta_{TH}$ ("H on Bob's coin"), her updated prevision is $\lambda_0 = 1$, that is the solution of

$$0 = L((q - \lambda_0)\pi) = -\lambda_0 z_{001} - \lambda_0 z_{100} + z_{011} + z_{101} + z_{110} + z_{200}$$

  - *Anne instantaneously knows that Bob's toss lead to H for him*
- The same holds in all other cases, hence the two coins are totally "correlated".
- Classical correlation can be explained by a common cause, or correlated "elements of reality". This is not the case in Bernstein's Posiland. Indeed, the marginal operators (states) satisfy

$$\begin{aligned} L(\theta_{H_2}) &= L(\theta_{H_1H_2} + \theta_{T_1H_2}) &&= z_{100} + z_{010} = \tfrac{1}{2}, \\ L(\theta_{H_1}) &= L(\theta_{H_1H_2} + \theta_{H_1T_2}) &&= z_{100} + z_{001} = \tfrac{1}{2}, \\ L(\theta_{H_2}^2) &= L((\theta_{H_1H_2} + \theta_{T_1H_2})^2) &&= z_{200} + 2z_{110} + z_{020} = \tfrac{1}{3}, \\ L(\theta_{H_1}^2) &:= L((\theta_{H_1H_2} + \theta_{H_1T_2})^2) &&= z_{200} + 2z_{101} + z_{002} = \tfrac{1}{3}. \end{aligned}$$

- Example of classical correlation model compatible with the marginal moments above is the mixture of atomic charges

$$p(\theta) = \frac{1}{2}\delta_{\left\{\begin{bmatrix} \frac{1}{6}(3-\sqrt{3}) \\ 0 \\ 0 \\ \frac{1}{6}(3+\sqrt{3}) \end{bmatrix}\right\}}(\theta) + \frac{1}{2}\delta_{\left\{\begin{bmatrix} \frac{1}{6}(3+\sqrt{3}) \\ 0 \\ 0 \\ \frac{1}{6}(3-\sqrt{3}) \end{bmatrix}\right\}}(\theta).$$

- However, this model (or any other) can *never* satisfy the moment constrains given by state L.
- We have **entanglement**.

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