On Minimum Elementary-triplet Bases for Independence Relations

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July 2019

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Probabilistic independence relations

A set of triplets $\langle \mathbf{A}, \mathbf{B} \mid \mathbf{C} \rangle$ with $\mathbf{A}, \mathbf{B}, \mathbf{C} \subset \mathbf{V}$, where a triplet $\langle \mathbf{A}, \mathbf{B} \mid \mathbf{C} \rangle$ captures that

$$\Pr(\mathbf{A}, \mathbf{B} \mid \mathbf{C}) = \Pr(\mathbf{A} \mid \mathbf{C}) \cdot \Pr(\mathbf{B} \mid \mathbf{C})$$

for all possible value combinations of $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

Not any subset of all possible triplets $\mathbf{V}^{(3)}$ is a probabilistic independence relation. For example, since $\langle \mathbf{A}, \mathbf{B} \mid \mathbf{C} \rangle$ implies $\langle \mathbf{B}, \mathbf{A} \mid \mathbf{C} \rangle$, each probabilistic independence relation will either include none or both of these triplets.

Semi-graphoid axioms

G1: if $\langle \mathbf{A}, \mathbf{B} \mid \mathbf{C} \rangle$ then $\langle \mathbf{B}, \mathbf{A} \mid \mathbf{C} \rangle$

G2: if $\langle \mathbf{A}, \mathbf{BD} \mid \mathbf{C} \rangle$ then $\langle \mathbf{A}, \mathbf{B} \mid \mathbf{C} \rangle$

G3: if $\langle \mathbf{A}, \mathbf{BD} \mid \mathbf{C} \rangle$ then $\langle \mathbf{A}, \mathbf{B} \mid \mathbf{CD} \rangle$

G4: if $\langle \mathbf{A}, \mathbf{B} \mid \mathbf{CD} \rangle$ and $\langle \mathbf{A}, \mathbf{D} \mid \mathbf{C} \rangle$ then $\langle \mathbf{A}, \mathbf{BD} \mid \mathbf{C} \rangle$

A semi-graphoid independence relation is a subset of triplets $\overline{J} \subseteq \mathbf{V}^{(3)}$ that satisfies the above properties for all sets $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \subseteq \mathbf{V}$.

A semi-graphoid independence relation $\overline{J}$ can be inferred from a starting set of triplets J by repeatedly applying the semi-graphoid axioms.

Elementary triplets

Triplets of the form $\langle A, B \mid \mathbf{C} \rangle$.

A semi-graphoid relationship is fully captured by its elementary triplets.

Semi-graphoid axioms for elementary triplets:

E1: if $\langle A, B \mid \mathbf{C} \rangle$ then $\langle B, A \mid \mathbf{C} \rangle$
E2: if $\langle A, B \mid \mathbf{C}D \rangle$ and $\langle A, D \mid \mathbf{C} \rangle$ then $\langle A, B \mid \mathbf{C} \rangle$ and $\langle A, D \mid \mathbf{C}B \rangle$

Bases for semi-graphoid independence relations

- Dominant triplets. Any triplet of the independence relation can be derived from one triplet in the basis through axioms G1-G3.
- Elementary triplets.

For example:

$\bar{J}$	dominant basis	elementary basis
$\langle 1, 2 \mid \varnothing \rangle$	$\langle 1, \{2, 3\} \mid \varnothing \rangle$	$\langle 1, 2 \mid \varnothing \rangle$
$\langle 1, 2 \mid 3 \rangle$		$\langle 1, 2 \mid 3 \rangle$
$\langle 1, 3 \mid \varnothing \rangle$		$\langle 1, 3 \mid \varnothing \rangle$
$\langle 1, 3 \mid 2 \rangle$		$\langle 1, 3 \mid 2 \rangle$
$\langle 1, \{2, 3\} \mid \varnothing \rangle$		

Minimum elementary triplet bases

Redundant information in an elementary triplet basis.

E2: if $\langle A, B \mid \mathbf{C}D \rangle$ and $\langle A, D \mid \mathbf{C} \rangle$ then $\langle A, B \mid \mathbf{C} \rangle$ and $\langle A, D \mid \mathbf{C}B \rangle$

For example, the elementary triplet basis
$\{\langle 1, 2 \mid \varnothing \rangle, \langle 1, 2 \mid 3 \rangle, \langle 1, 3 \mid \varnothing \rangle, \langle 1, 3 \mid 2 \rangle\}$ can be reduced to
$\{\langle 1, 2 \mid \varnothing \rangle, \langle 1, 3 \mid 2 \rangle\}$

Minimally needed:

- All A, B-combinations present in the independence relation.
- All cardinalities of $\mathbf{C}$ present in the independence relation.

Nb.

- A minimum elementary triplet basis is not unique.
- One by one removal of triplets does not necessarily yield a minimum basis.

Minimum bases for singleton starting sets

The semi-graphoid closure of the triplet

$$\langle \{A_1, \ldots, A_n\}, \{B_1, \ldots, B_m\} \mid \mathbf{C} \rangle$$

is also represented bij the elementary triplets

$$\left\{ \langle A_i, B_j \mid \mathbf{A} \backslash \{A_i, \ldots A_n\} \cup \mathbf{B} \backslash \{B_j, \ldots B_m\} \cup \mathbf{C} \, \rangle \, \middle| \, i = 1, \ldots, n, j = 1, \ldots, m \right\}$$

For example, the semi-graphoid closure of $\langle \{1, 2\}, \{3, 4\} \mid \varnothing \rangle$ is represented by the elementary triplets

$$\{\langle 1, 3 \mid \varnothing \rangle, \langle 1, 4 \mid 3 \rangle, \langle 2, 3 \mid 1 \rangle, \langle 2, 4 \mid \{1, 3\} \rangle\}$$

This implies that the semi-graphoid closure of $\langle \mathbf{A}, \mathbf{B} \mid \mathbf{C} \rangle$ can be represented by $|\mathbf{A}| \cdot |\mathbf{B}|$ elementary triplets.

A few questions

- How efficient are minimum elementary triplet bases compared to dominant triplet bases?
- How to compute a minimum basis efficiently?
- Is one by one removal of triplets a good heuristic?