Belief models

A very general theory of aggregation

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- Informativeness

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My plan is to show how far we can get with just these abstract ideas.

Introduction (again)

The very general theory of "Belief Models"[1] provides a neat generalisation of (part of) AGM belief revision theory.

[1] Gert de Cooman. "Belief models: An order-theoretic investigation". *Annals of Mathematics and Artificial Intelligence* 45 (2005), pp. 5–34

The very general theory of "Belief Models"[1] provides a neat generalisation of (part of) AGM belief revision theory.

My plan is to show that the same sort of generalisation can be applied to "merging operators"[2] for aggregating (propositional) knowledge bases.

[1] Gert de Cooman. "Belief models: An order-theoretic investigation". *Annals of Mathematics and Artificial Intelligence* 45 (2005), pp. 5–34

[2] Sébastien Konieczny and Ramón Pino Pérez. "Merging Information Under Constraints: A Logical Framework". *Journal of Logic and Computation* 12.5 (2002), pp. 773–808

Belief models

The recipe

- AGM expansion
- Merging operators

Cooking up aggregation rules

Some facts about sets of sentences

Consider the structure of sets of sentences of a propositional logic.

Ordering Sets of sentences are (partially) ordered by the subset relation.

Lattice structure For any pair of sets of sentences A, B, there is a set of sentences that is the least upper bound $A \vee B$, and another that is greatest lower bound $A \wedge B$.

Coherent substructure Some sets of sentences have the further property of being logically consistent and closed under consequence. Intersections of such sets also have this property.

Top The set of all sentences – the top of the ordering – is not coherent.

Some facts about lower previsions

Ordering Lower previsions are partially ordered by pointwise dominance. $P \preceq P'$ iff for all X, $P(X) \leq P'(X)$.

Lattice structure For any pair of lower previsions, there is a lower prevision that is the least upper bound and another that is the greatest lower bound.

Coherent substructure Some lower previsions have the further property of being coherent: they avoid sure loss. Pointwise minima of such lower previsions share this property.

Top The lower prevision that assigns ∞ to all gambles – the top of the structure – is not coherent.

Let $\mathbf{S}$ be a set of *belief models*, partially ordered by $\preceq$ (read as "is less informative than"), such that $\langle \mathbf{S}, \preceq \rangle$ is a complete lattice.

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$\langle \mathbf{S}, \mathbf{C}, \preceq \rangle$ is called a *belief structure*.

Let $\mathbf{M} = \{m \in \mathbf{C} : \text{For all } c \in \mathbf{C}, m \preceq c \Rightarrow m = c\}$

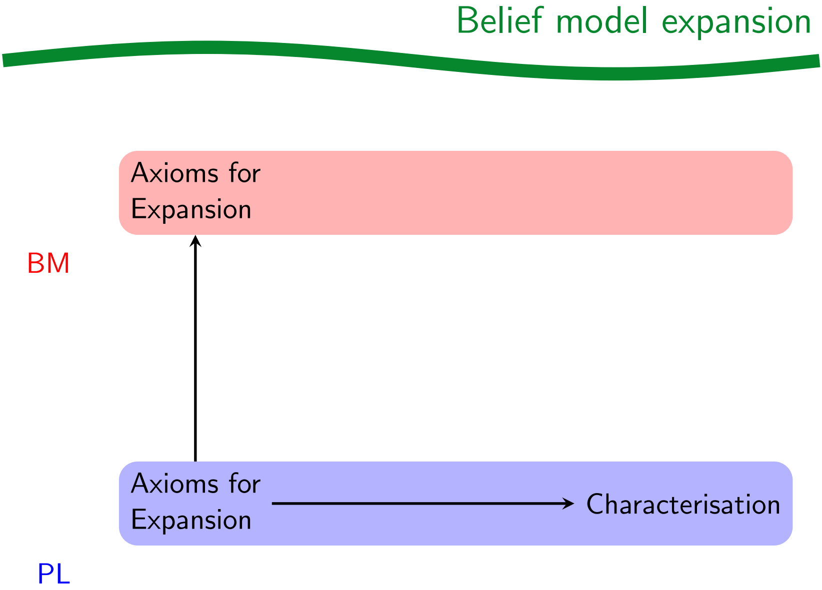
Examples of belief structures

- Propositional logic (with $\subseteq$, and consistent sets closed under consequence)
- Lower previsions (with pointwise dominance and closed convex credal sets)
- Modal logics and other nonstandard logics with well-behaved consequence operator
- Ranking functions
- Sets of desirable gambles, choice functions...
- Preference relations, comparative confidence relations?

Belief model expansion

Axioms for Expansion → Characterisation

PL


Belief model expansion
Axioms for
Expansion
BM
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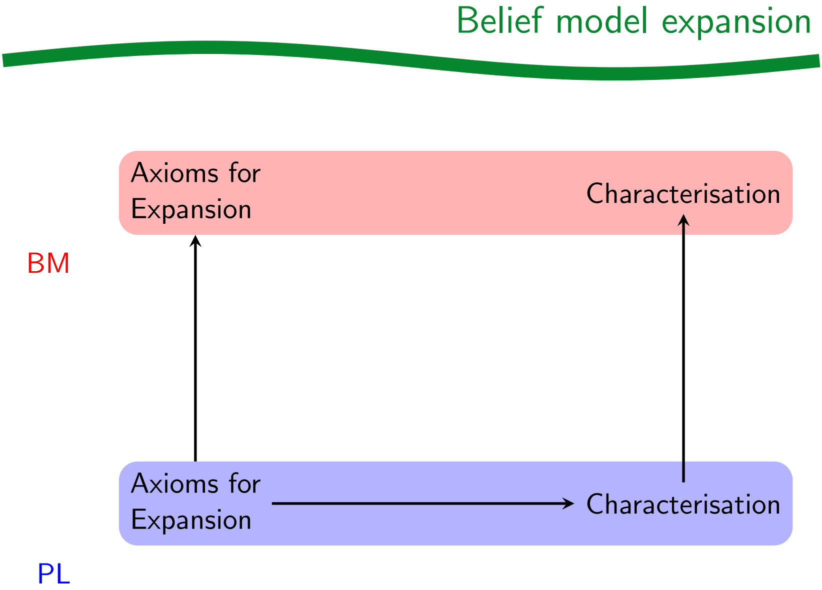
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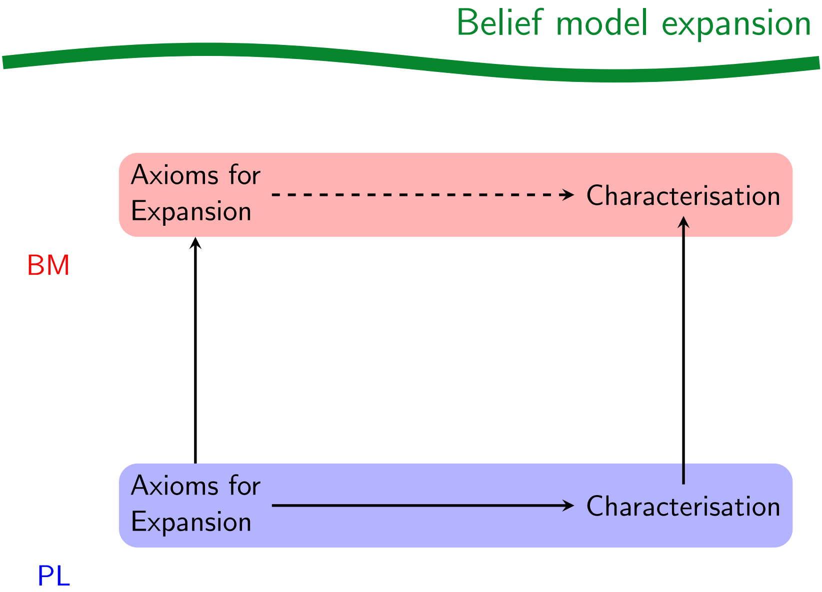

Belief model expansion
Axioms for Expansion
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The recipe

This recipe is quite generalisable: take a result framed in the theory of propositional logic, and (if you're lucky) it will also hold in some version of the belief models framework.

Merge: the basic idea

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Consider a multiset Ψ of belief models.

We want a function Δ that maps Ψ to some belief set, subject to some constraints:

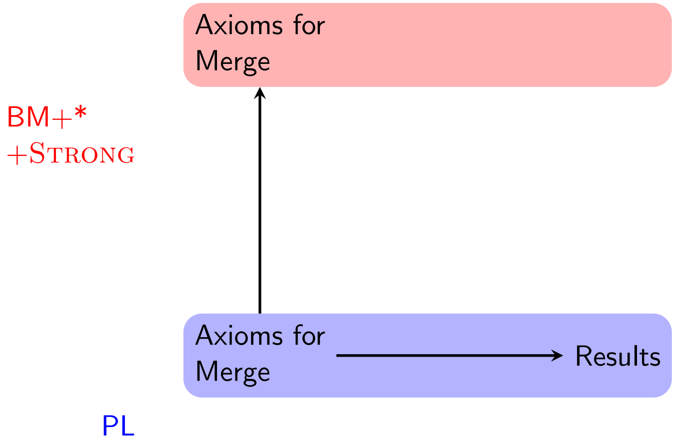
- It must satisfy some independent constraints (including consistency)
- It must be “as close” to the opinions of the members of Ψ as possible
- It must treat the different members of Ψ “fairly”

Belief model merging

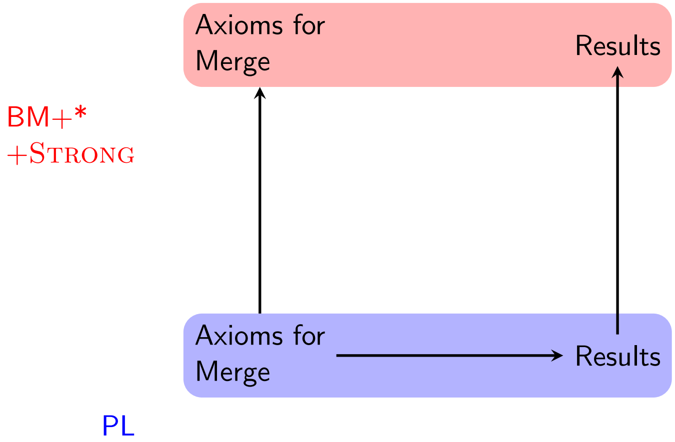
Axioms for Merge → Results

PL

Belief model merging



Belief model merging



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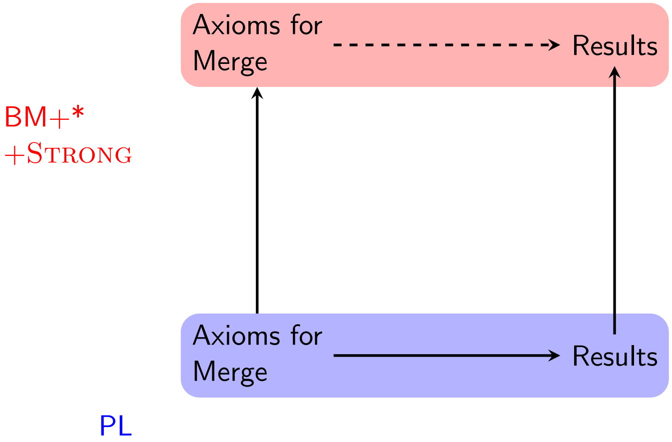

How to make a merging operator

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One way is to construct a Δ on the basis of a sort of "entrenchment relation" over $\mathbf{M}$.

Alternatively, you can construct a Δ using a "distance" over $\mathbf{M}$ and a method of aggregating distances.

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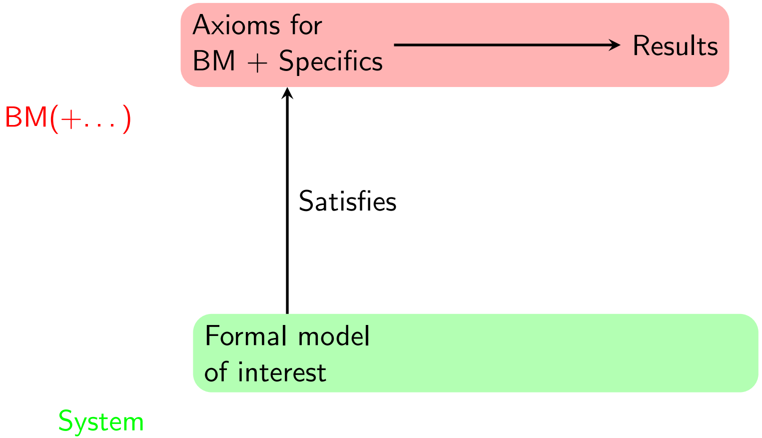
- Every "distance" and method of aggregating distances (…) yields a merging operator (…)

Belief models make new knowledge

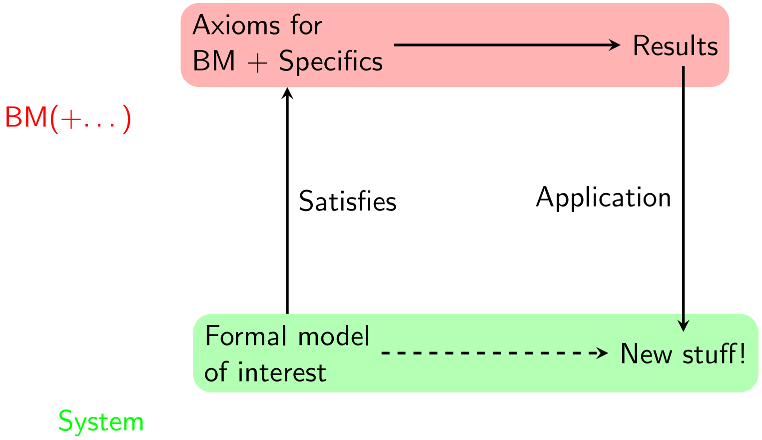
Axioms for BM + Specifics ⟶ Results

BM(+...)

Belief models make new knowledge



Belief models make new knowledge



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This procedure gives us a neat way to generate aggregation procedures for, e.g. lower previsions, ranking functions. . . , that satisfy certain desirable properties.

All we need to do is specify a distance between maximal coherent models, and a distance aggregation procedure.

For lower previsions (or closed convex credal sets) this amounts to specifying a distance over probability functions, and a aggregation function for real values.

Examples

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There is a generalisation of unweighted linear pooling that is an instance of this framework.

Summary

- Belief structures gives us a great way to easily import and generalise a bunch of work done using propositional logic
- More generally, it's remarkable how rich an interesting a theory of rational attitudes we can extract from just the concepts of Informativeness, Coherence and Closeness.

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Bonus material

- IP belief models
- AGM expansion, translated
- Merging operator
- Syncretic assignment
- Distance based merging

IP belief models

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Every coherent lower prevision, when restricted to events (indicator functions of sets of states) is a lower probability.

Every lower prevision has a non-empty set of linear previsions that dominates it. i.e. Each lower prevision has an associated closed convex set of probabilities.

Back

AGM

Call K_A^+ the expansion of K by (consistent) A.

1. K_A^+ is a belief set (i.e. closed under entailment and consistent)

Belief models

Call $E(b, c)$ the expansion operator for learning c on having beliefs b.

1. $E(b, c) \in \overline{\mathbf{C}}$

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Axioms

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2. $A \in K_A^+$
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5. If $K \subseteq H$ then $K_A^+ \subseteq H_A^+$

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2. $A \in K_A^+$
3. $K \subseteq K_A^+$
4. If $A \in K$ then $K_A^+ = K$
5. If $K \subseteq H$ then $K_A^+ \subseteq H_A^+$
6. For all K and A, K_A^+ is the smallest belief set satisfying the above conditions

Belief models

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2. $c \preceq E(b, c)$
3. $b \preceq E(b, c)$
4. If $c \preceq b$ then $E(b, c) = b$
5. If $b \preceq d$ then $E(b, c) \preceq E(d, c)$
6. $E(b, -)$ is the least informative of all the operators satisfying the above

Representation

AGM

If K_A^+ satisfies the above conditions, then $K_A^+ = Cn(K \cup \{A\})$.

Belief models

If E satisfies the above, then $E(b, c) = Cl_{\mathbf{S}}(\sup\{b, c\})$.

Back

Call $\Delta(\Psi, \mu)$ – or $\Delta_\mu(\Psi)$ – a *merging operator* if Ψ is a multiset of belief models, and μ is a belief model representing the constraints the aggregate belief must satisfy, and Δ satisfies:

- $\mu \preceq \Delta_\mu(\Psi)$
- If μ is consistent then $\Delta_\mu(\Psi)$ is consistent
- If $\bigvee \Psi \vee \mu$ is consistent then $\Delta_\mu(\Psi) = \bigvee \Psi \vee \mu$
- If $\mu \preceq \phi_1$ and $\mu \preceq \phi_2$ then $\Delta_\mu(\phi_1 \sqcup \phi_2) \vee \phi_1$ is consistent if and only if $\Delta_\mu(\phi_1 \sqcup \phi_2) \vee \phi_2$
- $\Delta_\mu(\Psi_1 \sqcup \Psi_2) \preceq \Delta_\mu(\Psi_1) \vee \Delta_\mu(\Psi_2)$
- If $\Delta_\mu(\Psi) \vee \Delta_\mu(\Psi_2)$ is consistent then, $\Delta_\mu(\Psi_1) \vee \Delta_\mu(\Psi_2) \preceq \Delta_\mu(\Psi_1 \sqcup \Psi_2)$
- $\Delta_{\mu_1 \vee \mu_2}(\psi) \preceq \Delta_{\mu_1}(\Psi) \vee \mu_2$
- If $\Delta_{\mu_1}(\Psi) \vee \mu_2$ is consistent then $\Delta_{\mu_1}(\Psi) \vee \mu_2 \preceq \Delta_{\mu_1 \vee \mu_2}(\psi)$

Syncretic assignments

A *syncretic assignment* is an assignment of a total preorder $\trianglelefteq_\Psi$ to each multiset Ψ, such that:

- For each Ψ, $\trianglelefteq_\Psi$ is a total order on $\mathbf{M}$
- If $a \in M(\bigvee \Psi)$ and $b \in M(\bigvee \Psi)$ then $a \trianglelefteq_\Psi b$
- If $a \in M(\bigvee \Psi)$ but $b \notin M(\bigvee \Psi)$ then $a \triangleleft_\Psi b$
- For all $a \in M(\phi)$ there is some $b \in M(\phi')$ such that $b \trianglelefteq_{\phi \sqcup \phi'} a$
- If $a \trianglelefteq_{\Psi_1} b$ and $a \trianglelefteq_{\Psi_2} b$ then $a \trianglelefteq_{\Psi_1 \sqcup \Psi_2} b$
- If $a \triangleleft_{\Psi_1} b$ and $a \trianglelefteq_{\Psi_2} b$ then $a \triangleleft_{\Psi_1 \sqcup \Psi_2} b$
- $\trianglelefteq_\Psi$ is *smooth*, meaning for all μ, for all $m \in M(\mu)$, if m is not minimal with respect to $\trianglelefteq_\Psi$ then there is an $m' \in M(\mu)$ such that m' is minimal and $m' \triangleleft_\Psi m$.

Δ is a merging operator iff there is a syncretic assignment such that $\Delta_\mu(\Psi) = \inf_{\preceq} \min_{\trianglelefteq_\Psi} \{M(\mu)\}$.

Distance based merging

Distance:

- D maps pairs of maximal coherent belief models to real numbers
- $D(m, m') = D(m', m)$
- $D(m, m') = 0$ iff $m = m'$.

Aggregation:

- F takes a sequence of real numbers and outputs a real number
- If $x \leq y$ then $F(x_1, \ldots, x, \ldots, x_n) \leq F(x_1, \ldots, y, \ldots, x_n)$
- $F(x_1, \ldots, x_n) = 0$ iff $x_1 = \cdots = x_n = 0$
- For all $x \in \mathbb{R}$, $F(x) = x$
- For a permutation σ, $F(x_1, \ldots, x_n) = F(\sigma(x_1), \ldots, \sigma(x_n))$
- $F(x_1, \ldots, x_n) \leq F(y_1, \ldots, y_n) \Rightarrow F(x_1, \ldots, x_n, z) \leq F(y_1, \ldots, y_n, z)$
- $F(x_1, \ldots, x_n) \leq F(y_1, \ldots, y_n) \Leftarrow F(x_1, \ldots, x_n, z) \leq F(y_1, \ldots, y_n, z)$

Back