UNIVERSITÀ
DEGLI STUDI DI TRIESTE

Extending Nearly-Linear Models

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Outline

Motivations

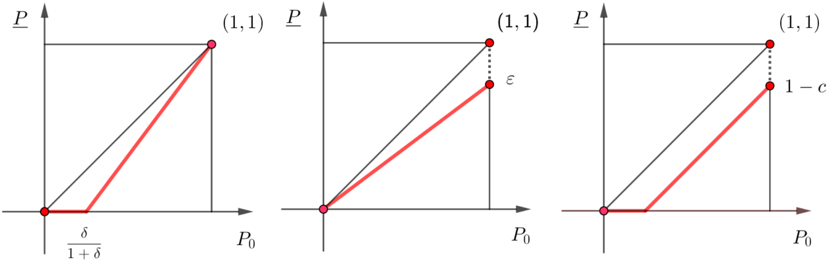
Nearly-Linear Models

- Definitions and basic properties
- Various types of natural extensions

Results postponed to the Poster Session

Motivations

- NL Models include several Neighbourhood Models:



Pari − Mutuel Model *ε − contamination Model* *Total Variation Model*

- NL Models may elicit various beliefs, even conflicting ones.

Nearly-Linear Models

NL Models are simple functions of a given probability P_0:

Definition (*Corsato, Pelessoni, Vicig, 2019*)

$\mu : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ is a Nearly-Linear (NL) imprecise probability if

- $\mu(\varnothing) = 0$, $\mu(\Omega) = 1$
- given P_0 on $\mathcal{A}(\mathbb{P})$, $a \in \mathbb{R}$, $b > 0$, $\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\varnothing, \Omega\}$,

$$\mu(A) = \min\{\max\{bP_0(A) + a, 0\}, 1\} = \max\{\min\{bP_0(A) + a, 1\}, 0\}.$$

We denote μ by $\mathrm{NL}(a, b)$.

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A NL μ is a **linear affine** transformation of P_0, **with barriers**.

Nearly-Linear Models - 2

- The family of NL imprecise probabilities is self-conjugate: if μ is $\mathrm{NL}(a,b)$, then μ^c is $\mathrm{NL}(c,b)$, with
$$c = 1 - (a + b).$$

Nearly-Linear Models - 2

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- $b + 2a \leq 1 \longrightarrow \underline{P}$ $\mathrm{NL}(a, b)$ 2-coherent (minimal consistency property).

Nearly-Linear Models - 3

We have *classified* 2-coherent Nearly-Linear Models into 3 subfamilies:

1. Vertical Barrier Model (VBM)
2. Horizontal Barrier Model (HBM)
3. Restricted Range Model (RRM)

and studied their *consistency properties* (*Corsato, Pelessoni, Vicig, 2019*).

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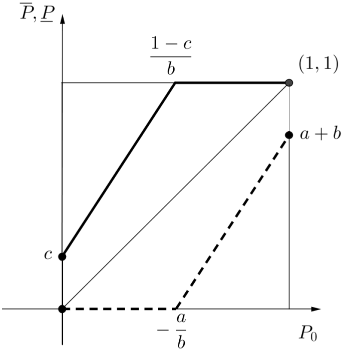
with $\underline{P}(\Omega) = 1$, $\overline{P}(\emptyset) = 0$.



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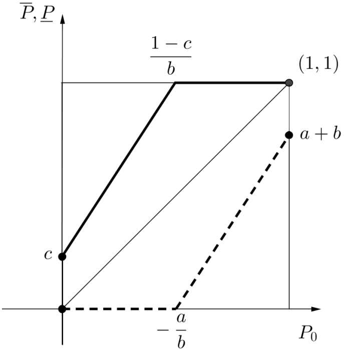

$\underline{P}$ is coherent and 2-monotone ($\overline{P}$ is coherent and 2-alternating).

VBM and natural extensions - 1

Proposition (VBM as a natural extension)

The lower probability in the VBM expression for $\underline{P}$,

$$\underline{Q}(A) = bP_0(A) + a, \quad \forall A \in \mathcal{A}(\mathbb{P}),$$

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A VBM is a *correction* of $\underline{Q}$ via natural extension.

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- If $a < 0$, for any $X \in \mathcal{L}(\mathbb{P})$ define

$$\tilde{x} = \sup\left\{x \in \mathbb{R} : P_0(X > x) \geq -\frac{a}{b}\right\}.$$

  Then

$$\underline{E}(X) = (a+b)\tilde{x} + \big(1-(a+b)\big)\inf X - bE^{P_0}\big((\tilde{x}-X)^+\big).$$

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Remark:

- $a < 0$ and $a + b = 1 \to$ PMM (*Walley, 1991*)
- $a = 0 \to \varepsilon$-contamination Model (*Walley, 1991*)

Horizontal Barrier Model (HBM)

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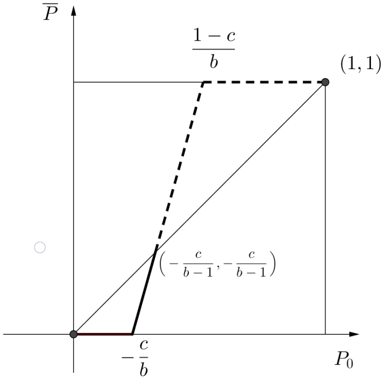

A HBM is generally only 2-coherent. It may avoid sure loss or be even coherent.

A selection of results for HBMs

- If $\mathbb{P}$ is finite, $\overline{P}$ in a HBM **avoids sure loss** iff $\sum_{\omega \in \mathbb{P}} \overline{P}(\omega) \geq 1$.

  Then its natural extension on $\mathcal{A}(\mathbb{P})$ is

  $$\overline{E}(A) = \min \Big\{ \sum_{\omega \in \mathbb{P}} \overline{P}(\omega), 1 \Big\}.$$

- $\overline{E}$ is also the natural extension of the probability interval $[0, \overline{P}(\omega)]_{\omega \in \mathbb{P}}$

  $\longrightarrow$ $\overline{E}$ is 2-monotone

  $\longrightarrow$ a HBM and a lower-vacuous probability interval avoiding sure loss are equivalent (*Troffaes, de Cooman, 2014*).

- If $\mathbb{P}$ is arbitrary, $\overline{P}$ avoids sure loss iff, for any *finite* partition $\mathbb{P}'$ coarser than $\mathbb{P}$,

  $$\sum_{\omega' \in \mathbb{P}'} \overline{P}(\omega') \geq 1.$$

Further (Poster Session) results

- Natural extensions of coherent HBMs
- Natural extensions of RRMs avoiding sure loss (further relationships with probability intervals)
- Interpretation of a VBM natural extension as a risk measure

Thank you...

...and see you at the Poster Session!