Imprecise Probabilities as a Semantics for Intuitive Probabilistic Reasoning

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Main references

- ISIPTA paper:

  H. Crane. (2019). Imprecise probabilities as a semantics for intuitive probabilistic reasoning.
  Researchers.One*, https://www.researchers.one/article/2018-08-8.*

- Further technical details:

  H. Crane. (2018). Logic of Probability and Conjecture.
  Researchers.One*, https://www.researchers.one/article/2018-08-5*

  H. Crane and I. Wilhelm. (2019). The Logic of Typicality. In Valia Allori (ed.), Statistical Mechanics and Scientific Explanation: Determinism, Indeterminism and Laws of Nature*, World Scientific.*
  Researchers.One*, https://www.researchers.one/article/2018-08-18*

Motivation

1. Some common statements of belief:
   - **Law**: I believe O.J. Simpson is not guilty of murder (beyond reasonable doubt) because the glove didn't fit.
   - **Mathematics**: I believe Goldbach's conjecture is probably true because it has been verified for $> 4 \times 10^{18}$ special cases.[1]
   - **Science**: There will be a partial solar eclipse on June 24, 2112 because that's what the laws of physics and relevant theories of planetary motion predict.[2]
   - **Common Sense**: It's safe to cross the street because there is no car within 200 yards.

2. These statements
   - rely on **intuition** about when it is reasonable to believe something,
   - involve **probabilistic judgment**, i.e., a judgment about what is 'probably true' in light of evidence, and
   - give **reasons** to justify the main claim.
3. They do *not*
   - convey a quantitative **degree of belief** about the claims.
4. All of these statements fall under heading of *intuitive probabilistic reasoning* (IPR).

[1] Goldbach's conjecture: every even integer greater than 3 is the sum of 2 primes, e.g., $3 + 1 = 4$, $3 + 3 = 6$, $5 + 3 = 8$, ...

[2] https://eclipse.gsfc.nasa.gov/SEcat5/SE2101-2200.html

Formalism of Intuitive Probabilistic Reasoning

- All of the previous statements have the form '*A* because *a*' where

  A is a claim (something I believe).

  a is a reason (justification) for the claim.

- These statements
  - convey subjective beliefs as well as provide an external qualification of that belief;
  - do not have the form of Bayesian credences (or quantitative 'beliefs' more generally).

- Main content of belief:
  - Bayes/probabilism: the *degree* of belief.
  - IPR: the *reason* for believing.

Main goal: *Formalize this process of reasoning.*
ISIPTA paper: *Show formal relationship between IPR and sets of probabilities.*

Formalism for IPR (Overview)

Formal:

- Syntax: Martin-Löf type theory (MLTT)
- Semantics: Homotopy type theory (HoTT)
- Extra structure: A type former **Bel** on top of usual rules of MLTT

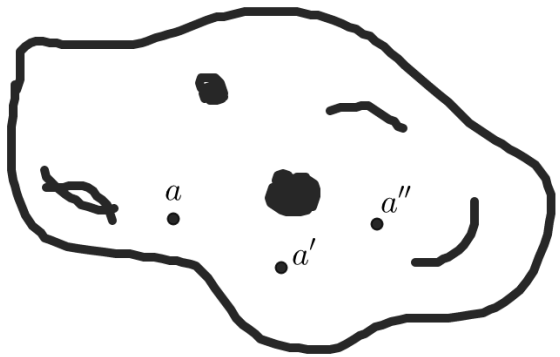
Pre-formal:

- Syntax: Rules for expressing judgments of the form '*A* because *a*'.
- Semantics: Subjective judgments reflect agent's subjective state of mind (credal state, context).
- Extra structure: Expressions about uncertain claims ('Probably *A* because *a*').

Basic Objects

Two basic objects:

IPR	Notation	Object (MLTT)	Formal (Set Theory)
claim	A	type	set
reason/justification	a	term	element



A : **Claim** $\leftrightarrow$ $A \equiv$ {Set of all ways to verify the claim}

$a, a', a'' \in A$ $\rightarrow$ pieces of evidence or justifications for the claim 'A'.

Contexts and Judgments

Judgments in MLTT:

Judgment	MLTT	IPR
A : **Type**	A is a type	A is a claim
$a : A$	a is a term of A	a is evidence for A

Contexts:

Judgment	MLTT	IPR
Δ **ctx**	Δ is a context	Δ is a state of mind, frame of reference

Role of context:

- All judgments are of the form above, and are made relative to a context:

$$\begin{array}{ccc} \Delta & \vdash & \mathcal{J} \\ \text{Context} & \vdash & \text{Judgment} \end{array}$$

- For example,
  $\Delta \vdash A$: **Claim** asserts that A is a meaningful/well-defined claim in context Δ.
  $\Delta \vdash a : A$ asserts that A holds because of a in context Δ.

Illustration

Full statement:

> *I believe that today is July 5, because yesterday someone on the bus said that it was July 4, and the day after July 4 is July 5.*

Formally:

Syntax	Meaning
A	Today is July 5.
$\mathbf{Bel}(A)$	Belief that A holds.
a	Claim by person on bus and implication that July 5 follows July 4.
$\Delta \vdash a : \mathbf{Bel}(A)$	Belief (from point of view Δ) that today is July 5 because of a.

Question

Is this a logical inference?

Approach:

*Devise a formal system for such inferences by introducing a new belief type (**Bel**) on top of existing machinery of MLTT/HoTT.*

Connection to Imprecise Probability

- Represent agent's state of mind by probability space $\Delta \equiv (\Omega, \mathcal{F}, P)$.
- $\mathcal{F}$: algebra of all 'meaningful claims' (i.e., claims about which agent has a credence).
- P: agent's credence function.

(Lockean Thesis)

Agent asserts belief in A just in case $P(A) \geq t$, for $1/2 < t \leq 1$ (Lockean threshold).

- For A : **Claim**, Δ can be characterized by

$$\Delta \subseteq \mathcal{F}_A := \{(\Omega, \mathcal{S}, \mu) \mid A \in \mathcal{S}\}$$
$$\Delta \subseteq P_A := \{(\Omega, \mathcal{S}, \mu) \mid \mu(A) = 1\}$$
$$\Delta \subseteq P_{\mathbf{Bel}(A)} := \{(\Omega, \mathcal{S}, \mu) \mid \mu(A) \geq t\}.$$

Main idea:

- $\mathcal{F}_A$: frames of mind for which A is a meaningful claim (i.e., assigns credence).
- $\mathcal{P}_A$: frames of mind for which A is true (i.e., assigns maximal credence).
- $\mathcal{P}_{\mathbf{Bel}(A)}$: frames of mind for which A is believed to be true (i.e., assigns sufficiently high credence).

Lockeans follow IPR

Theorem

Lockean semantics are sound for IPR.

Translation of syntax:

IPR/MLTT	Probability	Interpretation
Δ **ctx**	$\Delta \subseteq \mathcal{P}(\Omega)$	An agent's frame of reference is a subset of probability spaces
$\Delta \vdash A$: **Claim**	$\Delta \subseteq \mathcal{F}_A$	A claim is meaningful from viewpoint Δ if every element of Δ assigns A a credence.
$\Delta \vdash a : A$	$\Delta \subseteq P_A$	A is true from viewpoint Δ if every element of Δ assigns maximal credence to A.
$\Delta \vdash a$: **Bel**(A)	$\Delta \subseteq P_{\mathbf{Bel}(A)}$	A is believed to be true from viewpoint Δ if every element of Δ assigns high credence ($\geq t$) to A.

Conclusion

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