Monte Carlo Estimation for Imprecise Probabilities

Basic Properties

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Setting

Monte Carlo

$$\frac{1}{n}\sum_{k=1}^{n} f(X_k^P) \approx \mathrm{E}^P(f)$$

Imprecise Probability

$$\mathscr{P} = \{P_t \colon t \in T\}$$

$$\underline{\mathrm{E}}^{\mathscr{P}}(f) = \inf_t \mathrm{E}^{P_t}(f)$$

Estimators for lower expectations

- $E^{P_1}(f)$
- $E^{P_2}(f)$

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Infinite set of probability measures

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$$\underline{\mathrm{E}}^{\mathscr{P}}(f) \overset{?}{\approx} \inf_t \sum_{k=1}^{n} f_t\left(X_k^P\right) = \underline{\hat{\mathrm{E}}}$$

Bias

negative

$\underline{E}(f)$

$E(\hat{\underline{E}})$

unbiased

$E(\hat{\underline{E}}) = \underline{E}(f)$

positive

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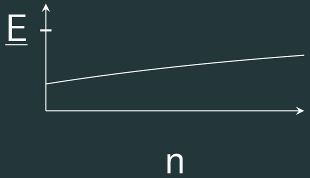
can only get closer



unbiased

$E(\underline{\hat{E}}) = \underline{E}(f)$

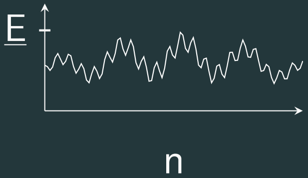
constant

positive

$E(\underline{\hat{E}})$

$\underline{E}(f)$

it depends



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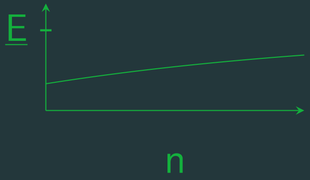
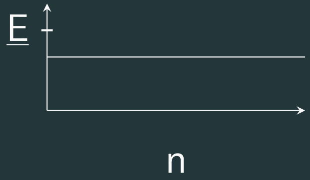


constant



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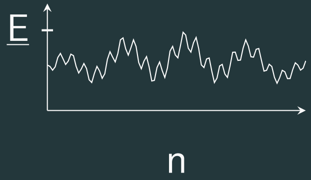

consistency

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1. In probability

$$\lim_{n\to\infty} P^{\infty}\left(\left|\hat{\underline{\mathsf{E}}}_n - \mathsf{E}(f)\right| > \epsilon\right) = 0$$

2. Almost surely

$$P^{\infty}\left(\lim_{n\to\infty} \hat{\underline{\mathsf{E}}}_n = \mathsf{E}(f)\right) = 0$$

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Consistency

$$\inf_t \sum_{k=0}^{n} f_t\left(X_k^P\right) \overset{?}{\to} \inf_t \mathrm{E}^{P_t}(f) = \underline{\mathrm{E}}^{\mathscr{P}}(f) \qquad \text{as } n \to \infty$$

Consistency

$$\inf_t \sum_{k=0}^{n} f_t\left(X_k^P\right) \xrightarrow{?} \inf_t \mathsf{E}^{P_t}(f) = \underline{\mathsf{E}}^{\mathscr{P}}(f) \qquad \text{as } n \to \infty$$

$$\Uparrow$$

$$\sup_t \left| \sum_{k=0}^{n} f_t\left(X_k^P\right) - \mathsf{E}^{P_t}(f) \right| \xrightarrow{?} 0 \qquad \text{as } n \to \infty$$

Consistency

When is this the case?

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- restrictions on size of T

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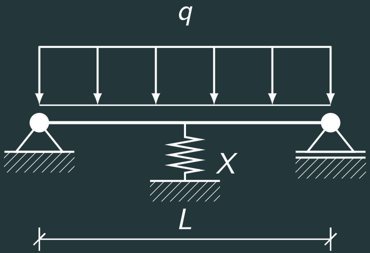
finite T	$\mathbb{R}^n \supset T$ compact $p_t(x)$ cont. diff. in (x, t) $\mathrm{E}^P(\sup_{t \in T} p_t) < +\infty$	$\mathbb{R}^n \supset T$ bounded $\|\nabla_t p_t(x)\| < F(x)$	for all $\epsilon > 0$: T has a finite ϵ-cover $\lvert p_t(x) - p_s(x) \rvert \leqslant d(s, t) F(x)$

← easier but more restrictive — more general but more complex →

Practical Example



$$\overline{P}(g(X) \leqslant 0) = 1 - \underline{\mathsf{E}}^{\mathscr{P}}\left(\mathbb{I}_{\{g(X)>0\}}\right)$$

Fetz, T., & Oberguggenberger, M. (2016). Imprecise random variables, random sets, and Monte Carlo simulation.

See you at my poster.