An Axiomatic Utility Theory for Dempster-Shafer Belief Functions

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Introduction

- Main goal is to propose an axiomatic utility theory for D-S belief function lotteries similar to vN-M's axiomatic framework for probabilistic lotteries.
- D-S theory consists of representations (basic probability assignments, belief, plausibility, commonality, credal sets) + Dempster's combination rule + marginalization rule.
- Representations are also used in other theories, e.g., in the imprecise probability community, credal sets are used with Fagin-Halpern combination rule.
- Our axiomatic utility theory is designed for the D-S theory.
- Therefore, Dempster's combination must be an integral part of our theory.

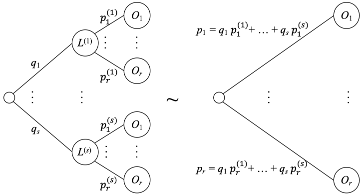
vN-M's Utility Theory

- Let $\mathbf{O} = (O_1, \ldots, O_r)$ denote a finite set of outcomes.
- Let $\mathbf{p} = (p_1, \ldots, p_r)$ denote a probability mass function (PMF) on $\mathbf{O}$, i.e., $p_i \geq 0$ for $i = 1, \ldots, r$, and $\sum_{i=1}^{r} p_i = 1$.
- We call $L = [\mathbf{O}, \mathbf{p}]$ a probabilistic lottery on $\mathbf{O}$. We assume that L will result in one outcome O_i (with prob. p_i), and it is not repeated.
- We are concerned with a decision maker (DM) who has preferences on $\mathcal{L}$, the set of all lotteries on $\mathbf{O}$.
- We write $L \succ L'$ if the DM prefers L to L', $L \sim L'$ if the DM is indifferent between L and L', and $L \succsim L'$ is the DM either prefers L to L' or is indifferent between the two.
- Our task is to find a real-valued utility function $u : \mathcal{L} \to \mathbb{R}$ such that if $L \succ L'$, then $u(L) > u(L')$, and if $L \sim L'$, then $u(L) = u(L')$.
- There are several axiomatizations of vN-M's utility theory by Herstein-Milnor [1953], Hausner [1954], Luce-Raiffa [1957], Jensen [1967], Fishburn [1982], etc. We will describe the one by Luce-Raiffa [1957].

vN-M's Utility Theory

- **Assumption** $1p$ (ordering of outcomes). For any two outcomes O_i and O_j, either $O_i \succsim O_j$ or $O_j \succsim O_i$. Also, if $O_i \succsim O_j$ and $O_j \succsim O_k$, then $O_i \succsim O_k$. Thus, ordering $\succsim$ over **O** is complete and transitive.
- Given Assumption $1p$, we can label the outcomes so that $O_1 \succsim O_2 \succsim \ldots \succsim O_r$.
- To avoid trivialities, we assume $O_1 \succ O_r$.

vN-M's Utility Theory



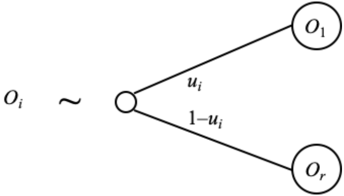
- **Assumption** $2p$ (reduction of compound lotteries). Any compound lottery $[\mathbf{L}, \mathbf{q}]$ (where $\mathbf{L} = (L^{(1)}, \ldots, L^{(s)})$, and $L^{(i)} = [\mathbf{O}, \mathbf{p}^{(i)}]$) is indifferent to a simple (non-compound) lottery $[\mathbf{O}, \mathbf{p}]$, where

$$p_i = q_1\, p_i^{(1)} + \ldots + q_s\, p_i^{(s)} \tag{1}$$

- PMF $\mathbf{p}^{(i)}$ is a conditional PMF for $\mathbf{O}$ given that lottery $L^{(i)}$ is realized in the first stage.
- The PMF $\mathbf{p} = (P(\mathbf{L}) \otimes P(\mathbf{O}|\mathbf{L}))^{\downarrow \mathbf{O}}$ is the marginal of the joint PMF for $\mathbf{O}$.

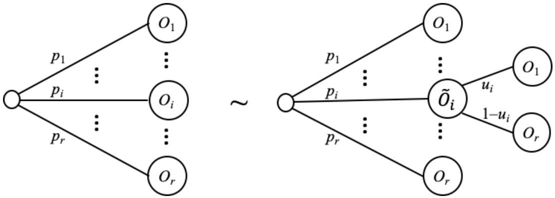
vN-M's Utility Theory

- A lottery $[(O_1, O_r), (u, 1-u)]$ with only two outcomes O_1 and O_r, with PMF $(u, 1-u)$ is called a reference lottery. Let $\mathbf{O}_2$ denote (O_1, O_r).
- **Assumption** $3p$ (continuity) Each outcome O_i is indifferent to a reference lottery $[\mathbf{O}_2, (u_i, 1-u_i)]$ for some $0 \leq u_i \leq 1$, i.e., $O_i \sim \widetilde{O}_i$, where $\widetilde{O}_i = [\mathbf{O}_2, (u_i, 1-u_i)]$.
- Notice that $u_1 = 1$, $u_r = 0$, and $0 \leq u_i \leq 1$ for $i = 2, \ldots, r-1$. $u_2, \ldots, u_{r-1}$ need to be assessed by the DM, and the assessments describe the risk attitude of the DM.



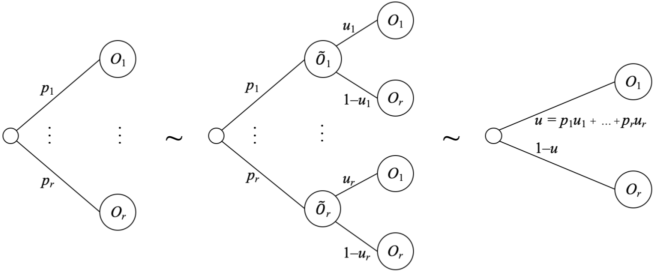
vN-M's Utility Theory

- **Assumption** $4p$ (completeness and transitivity) The preference relation $\succsim$ for lotteries in $\mathcal{L}$ is complete and transitive.
- Assumption $4p$ generalizes Assumption $1p$ for outcomes, which can be regarded as degenerate lotteries.
- **Assumption** $5p$ (substitutability) In any lottery $L = [\mathbf{O}, \mathbf{p}]$, if we substitute an outcome O_i by the reference lottery $\widetilde{O}_i = [\mathbf{O}_2, (u_i, 1 - u_i)]$ that is indifferent to O_i, then the result is a compound lottery that is indifferent to L.



vN-M's Utility Theory

- From Assumptions $1p - 5p$, given any lottery $L = [\mathbf{O}, \mathbf{p}]$, it is possible to find a reference lottery that is indifferent to L:



vN-M's Utility Theory

- **Assumption** $6p$ (monotonicity) Suppose $L = [\mathbf{O}_2, (p, 1-p)]$ and $L' = [\mathbf{O}_2, (p', 1-p')]$. Then $L \succsim L'$ if and only if $p \geq p'$.
- Assumption $6p$ allows us to define $u(L)$ as the utility of O_1 in an indifferent reference lottery. And as argued in the previous slide, we can always find a reference lottery that is indifferent to L.

vN-M's Utility Theory

Theorem (vN-M representation theorem)

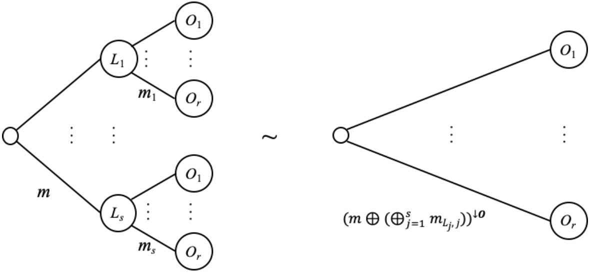
If the preference relation $\succsim$ *on* $\mathcal{L}$ *satisfies Assumptions* $1p - 6p$*, then there are numbers* u_i *associated with outcomes* O_i *for* $i = 1, \ldots, r$*, such that for any two lotteries* $L = [\mathbf{O}, \mathbf{p}]$*, and* $L' = [\mathbf{O}, \mathbf{p}']$*,* $L \succsim L'$ *if and only if*

$$\sum_{i=1}^{r} p_i \, u_i \geq \sum_{i=1}^{r} p'_i \, u_i$$

Thus, for $L = [\mathbf{O}, \mathbf{p}]$*, we can define* $u(L) = \sum_{i=1}^{r} p_i \, u_i$*, where* $u_i = u(O_i)$*. Also, such a linear utility function is unique up to a positive affine transformation, i.e., if* $u'_i = a \, u_i + b$*, where* $a > 0$ *and* b *are real constants, then* $u(L) = \sum_{i=1}^{r} p_i \, u'_i$ *is also qualifies as a utility function.*

A New Utility Theory for D-S Belief Functions

- **Assumption** $2b$ (reduction of compound lotteries) Suppose $[\mathbf{L}, m]$ is a bf compound lottery, where $\mathbf{L} = \{L_1, \ldots, L_s\}$, m is a BPA for $\mathbf{L}$, $L_j = [\mathbf{O}, m_j]$ is a bf lottery on $\mathbf{O}$, and m_j is a conditional BPA for $\mathbf{O}$ given L_j, for $j = 1, \ldots, s$. Then, $[\mathbf{L}, m] \sim [\mathbf{O}, m']$, where $m' = (m \oplus (\bigoplus_{j=1}^{s} m_{L_j,j}))^{\downarrow \mathbf{O}}$, and $m_{L_j,j}$ is a BPA for $(\mathbf{L}, \mathbf{O})$ obtained from BPA m_j for $\mathbf{O}$ by conditional embedding, for $j = 1, \ldots, s$.

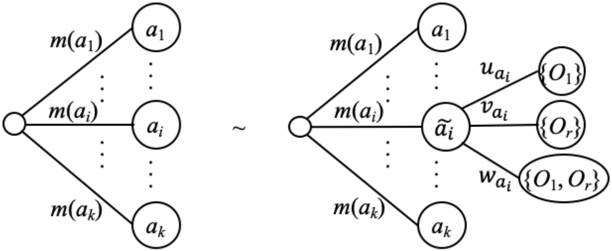


A New Utility Theory for D-S Belief Functions

- We define a reference bf lottery $[\mathbf{O}_2, m]$, where m is a BPA for $\mathbf{O}_2 = \{O_1, O_r\}$.
- **Assumption** $3b$ (continuity) Suppose $[\mathbf{O}, m]$ is a bf lottery derived from some BPA m'. Each focal element a of m (considered as a deterministic bf lottery) is indifferent to a bf reference lottery $[\mathbf{O}_2, m_{\mathsf{a}}]$ such that $m_{\mathsf{a}}(\{O_1\}) = u_{\mathsf{a}}$, $m_{\mathsf{a}}(\{O_r\}) = v_{\mathsf{a}}$, and $m_{\mathsf{a}}(\{O_1, O_r\}) = w_{\mathsf{a}}$, for some $u_{\mathsf{a}}, v_{\mathsf{a}}, w_{\mathsf{a}} \geq 0$, and $u_{\mathsf{a}} + v_{\mathsf{a}} + u_{\mathsf{a}} = 1$. Furthermore, $w_{\mathsf{a}} = 0$ if a $= \{O_i\}$ is a singleton focal set of m.
- Assumption 3b is a generalization of Assumption 3p.

A New Utility Theory for D-S Belief Functions

- **Assumption** $4b$ (reflexive and transitive) The preference relation $\succsim$ for $\mathcal{L}_{bf}$ is reflexive and transitive.
- In comparison with Assumption $4p$, we do not assume that $\succsim$ is complete. It is neither descriptive nor normative, and consistent with D-S theory philosophy of incomplete knowledge.
- **Assumption** $5b$ (substitutability) In any bf lottery $L = [\mathbf{O}, m]$, if we substitute a focal element a_i of m by an equally preferred bf reference lottery $\widetilde{\mathsf{a}_i} = [\mathbf{O}_2, m_{\mathsf{a}_i}]$, then the result is a compound lottery that is indifferent to L.



A New Utility Theory for D-S Belief Functions

Theorem (Reducing a bf lottery to an indifferent bf reference lottery)

Under Assumptions $1b - 5b$, any bf lottery $L = [\boldsymbol{O}, m]$ with focal sets $\mathsf{a}_1, \ldots \mathsf{a}_k$ is indifferent to a bf reference lottery $\widetilde{L} = [\boldsymbol{O}_2, \widetilde{m}]$, such that

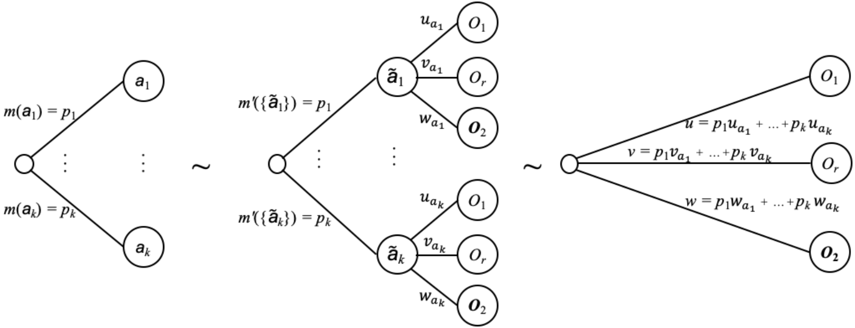
$$\widetilde{m}(\{O_1\}) = \sum_{i=1}^{k} m(\mathsf{a}_i)\, u_{\mathsf{a}_i}, \tag{2a}$$

$$\widetilde{m}(\{O_r\}) = \sum_{i=1}^{k} m(\mathsf{a}_i)\, v_{\mathsf{a}_i}, \quad \text{and} \tag{2b}$$

$$\widetilde{m}(\boldsymbol{O}_2) = \sum_{i=1}^{k} m(\mathsf{a}_i)\, w_{\mathsf{a}_i}, \tag{2c}$$

where u_{a_i}, v_{a_i}, and w_{a_i}, are the masses assigned, respectively, to $\{O_1\}$, $\{O_r\}$, and $\boldsymbol{O}_2$, by the bf reference lottery $\widetilde{\mathsf{a}}_i$ equivalent to a_i.

A New Utility Theory for D-S Belief Functions



A New Utility Theory for D-S Belief Functions

- **Assumption** $6b$ (monotonicity) Suppose $L = [\mathbf{O}_2, m]$ and $L' = [\mathbf{O}_2, m']$ are bf reference lotteries, with $m(\{O_1\}) = u$, $m(\mathbf{O}) = w$, $m'(\{O_1\}) = u'$, $m'(\mathbf{O}) = w'$. Then, $L \succsim L'$ if and only if $u \geq u'$ and $u + w \geq u' + w'$.
- Thus, $L \succsim L'$ if and only if $Bel_m(O_1) \geq Bel_{m'}(O_1)$ and $Pl_m(O_1) \geq Pl_{m'}(O_1)$, i.e., if and only if outcome O_1 is both more credible and more plausible under L than L'.
- The corresponding indifference relation is: $L \sim L'$ if and only if $u = u'$ and $w = w'$.
- It is clear that $\succsim$ as defined in Assumption $6b$ is reflexive and transitive.
- Thus, L and L' are *incomparable* if one of the intervals $[u, u + w]$ and $[u', u' + w']$ is strictly included in the other.

A New Utility Theory for D-S Belief Functions

Theorem (Interval-valued utility for bf lotteries)

Suppose $L = [\boldsymbol{O}, m]$ *and* $L' = [\boldsymbol{O}, m']$ *are bf lotteries on* $\boldsymbol{O}$*. If the preference relation* $\succsim$ *on* $\mathcal{L}_{bf}$ *satisfies Assumptions* $1b - 6b$*, then there are intervals* $[u_{a_i}, u_{a_i} + w_{a_i}]$ *associated with subsets* $a_i \in 2^{\boldsymbol{O}}$ *such that* $L \succsim L'$ *iff*

$$\sum_{a_i \in 2^{\boldsymbol{O}}} m(a_i)\, u_{a_i} \geq \sum_{a_i \in 2^{\boldsymbol{O}}} m'(a_i)\, u_{a_i}, \text{ and}$$

$$\sum_{a_i \in 2^{\boldsymbol{O}}} m(a_i)\, (u_{a_i} + w_{a_i}) \geq \sum_{a_i \in 2^{\boldsymbol{O}}} m'(a_i)\, (u_{a_i} + w_{a_i}).$$

Thus, for a bf lottery $L = [\boldsymbol{O}, m]$*, we can define* $u(L) = [u, u + w]$ *as an interval-valued utility of* L*, with* $u = \sum_{a_i \in 2^{\boldsymbol{O}}} m(a_i)\, u_{a_i}$ *and* $w = \sum_{a_i \in 2^{\boldsymbol{O}}} m(a_i)\, w_{a_i}$*. Also, such a utility function is unique up to a strictly increasing affine transformation.*

A New Utility Theory for D-S Belief Functions

- Our final assumption has no counterpart in the vN-M theory.
- **Assumption** $7b$ (consistency) Let $\mathsf{a} \subseteq \mathbf{O}$, and let $\underline{O}_{\mathsf{a}}$ and $\overline{O}_{\mathsf{a}}$ denote, respectively, the worst and the best outcome in a. Then we have

$$\mathsf{a} \succsim \underline{O}_{\mathsf{a}} \quad \text{and} \quad \overline{O}_{\mathsf{a}} \succsim \mathsf{a}.$$

- Assumptions $6b$ and $7b$ imply that, for any focal sets a of m, we have

$$u_{\mathsf{a}} \geq \min_{O_i \in \mathsf{a}} u_{\{O_i\}}, \quad \text{and} \quad u_{\mathsf{a}} + w_{\mathsf{a}} \leq \max_{O_i \in \mathsf{a}} u_{\{O_i\}}. \tag{3}$$

A New Utility Theory for D-S Belief Functions

- In the imprecise literature, we have lower and upper Choquet integrals defined as follows:

Definition (Choquet integrals)

Suppose we have a real-valued function $u : \mathbf{O} \to \mathbb{R}$. The lower and upper Choquet integrals of u with respect to BPA m for $\mathbf{O}$, denoted by $\underline{u}_m$ and $\overline{u}_m$, are defined as follows:

$$\underline{u}_m = \sum_{\mathsf{a} \in 2^{\mathbf{O}}} m(\mathsf{a}) \left(\min_{O_i \in \mathsf{a}} u(O_i) \right),$$

$$\overline{u}_m = \sum_{\mathsf{a} \in 2^{\mathbf{O}}} m(\mathsf{a}) \left(\max_{O_i \in \mathsf{a}} u(O_i) \right).$$

- Thus, we can regard the interval $[\underline{u}_m, \overline{u}_m]$ as an interval-valued utility of $[\mathbf{O}, m]$.

A New Utility Theory for D-S Belief Functions

- It follows from Theorem 2 and Assumption $7b$ that

$$\underline{u}_m \leq u \leq u + w \leq \overline{u}_m.$$

  where u and w are as in Theorem 3.
- Thus, the interval-valued utility defined in Theorem 3 is always included in the lower and upper Choquet integral interval-valued utility.

Summary & Conclusions

- We have proposed an axiomatic utility theory for D-S lotteries similar to vN-M's utility theory for probabilistic lotteries,
- The main difference is singleton outcomes are replaced by focal elements of m, probabilistic combination is replaced by Dempster's combination rule, and probabilistic marginalization is replaced by belief function marginalization.
- Our axiomatic theory is able to explain ambiguity attitude of human DMs that vN-M's utility theory cannot
- While there are several probabilistic decision theories that explain ambiguity-attitude of human DMs (Becker and Brownson 1964, Einhorn and Hogarth 1986, etc.), they are not justified by simple axioms similar to vN-M's or Savage's.