Coherent upper conditional previsions defined by Hausdorff outer measures for unbounded random variables

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Aim

In a metric space coherent upper conditional previsions defined by Hausdorff outer measures, are extended to the linear space of bounded and unbounded random variables with finite Choquet integral with respect to the Hausdorff outer and inner measures.

Motivations

Coherent linear conditional previsions and the Radon-Nikodym derivative

The necessity to propose a new tool to define coherent upper conditional previsions arises because they cannot be obtained as extensions of linear expectations defined, by the Radon-Nikodym derivative, in the axiomatic approach; it occurs because one of the defining properties of the Radon-Nikodym derivative, that is to be measurable with respect to the σ-field of the conditioning events, contradicts the following

Necessary condition for coherence

If for every B belongs to $\mathbf{B}$ $P(X|B)$ are coherent linear expectations and X is $\mathbf{B}$-measurable then $P(X|\mathbf{B}) = X$.

Motivations

Indifference between random variables with the same distribution

If coherent previsions are not continuous from below they preclude indifference between equivalent random variables as may occur for random variables with geometric distribution (Seidenfeld et al., 2009).

The model

Theorem

Let (Ω, d) be a metric space and let $\mathbf{B}$ be a partition of Ω. For every $B \in \mathbf{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s-dimensional outer measure. Let m be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$. Thus, for each $B \in \mathbf{B}$, the function defined on $\wp(B)$ by

$$\overline{P}(A|B) = \begin{cases} \frac{h^s(A \cap B)}{h^s(B)} & \text{if} \quad 0 < h^s(B) < +\infty \\ m_B & \text{if} \quad h^s(B) \in \{0, +\infty\} \end{cases}$$

is a coherent upper conditional probability.

The model

Restriction to the class of measurable sets

If $B \in$ **B** is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s the monotone set function μ_B^* defined for every $A \in \wp(B)$ by $\mu_B^*(A) = \frac{h^s(AB)}{h^s(B)}$ is a coherent upper conditional probability, which is submodular, continuous from below and such that its restriction to the σ-field of all μ_B^*-measurable sets is a Borel regular countably additive probability.

The model

The domain

Since the class of the bounded and unbounded random variables which admit Choquet integral is not a linear space, first it is proven that the class $L^*(B)$ of all random variables which have finite Choquet integral with respect to the coherent upper conditional probability μ_B^* and with respect to its conjugate lower conditional probability is a linear space.

The model

Theorem

Let (Ω, d) be a metric space and let **B** *be a partition of* Ω. *For* $B \in \mathbf{B}$ *denote by s the Hausdorff dimension of the conditioning event B and by* h^s *the Hausdorff s-dimensional outer measure. Let* m_B *be a 0-1 valued finitely additive, but not countably additive, probability on* $\wp(B)$. *Then for each* $B \in \mathbf{B}$ *the functional* $\overline{P}(X|B)$ *defined on the linear space* $L^*(B)$ *by*

$$\overline{P}(X|B) = \begin{cases} \frac{1}{h^s(B)} \int_B X dh^s & \text{if} \quad 0 < h^s(B) < +\infty \\ m_B & \text{if} \quad h^s(B) \in \{0, +\infty\} \end{cases}$$

is a coherent upper conditional prevision.

Main results

Monotone Convergence Theorem

Coherent upper conditional previsions are continuous from below and they satisfy the Monotone Convergence Theorem when the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension.

Main results

Disintegration property

If Ω is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension coherent upper prevision $\overline{P}$ satisfies the disintegration property $\overline{P}(X) = \overline{P}(\overline{P}(X|\mathbf{B}))$ on every non- null partition $\mathbf{B}$.

Main results

Relation with other coherent upper probabilities defined on $\wp(B)$

All monotone set functions on $\wp(B)$ which are submodular, continuous from below and which represent as Choquet integral a coherent upper conditional prevision defined on a linear lattice $\mathbf{F}$, agree on the set system of weak upper level sets $M = \{\{X \geq x\} \, | X \in \mathbf{F}, x \in \Re\}$, with the coherent upper conditional probability $\mu^*(A) = \frac{h^s(AB)}{h^s(B)}$ for $A \in \wp(B)$.