Markov chains under nonlinear expectation

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06. 07. 2019

ISIPTA 2019

Ghent, Belgium

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Pre-expectations are closely related to (monetary) risk measures introduced by Artzner et al. (1999), Delbaen (2000, 2002), see also Föllmer-Schied (2011) and upper/lower previsions by Walley (1991).

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- ...(pre-)expectations that are *continuous from above or below*, i.e.

$$\mathcal{E}(X_n) \searrow \mathcal{E}(X) \quad \text{or} \quad \mathcal{E}(X_n) \nearrow \mathcal{E}(X)$$

  for $(X_n)_{n\in\mathbb{N}} \subset M$ with $X_n \searrow X \in M$ or $X_n \nearrow X \in M$, respectively.

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where $\mathcal{P}$ is a nonempty set of finitely additive probability measures and $\alpha_{\mathbb{P}} \geq 0$ is a penalization for the model $\mathbb{P}$.
- If $\mathcal{E}$ is, additionally, continuous from above, the set $\mathcal{P}$ contains only countably additive probability measures.

Extension of pre-expectations (Denk-Kupper-N. (2018))

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1) Extension without continuity assumptions: For $X \in \mathscr{L}^\infty(\Omega, \mathscr{F})$, let

$$\hat{\mathcal{E}}(X) := \inf \left\{ \mathcal{E}(Y) \,\middle|\, Y \in M,\ Y \geq X \right\}.$$

- Inspired by Kantorovich's extension of positive linear functionals,
- Closely linked to the idea of superhedging ($\approx$ NFL),
- Preserves convexity and sublinearity,
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2) Extension if $\mathcal{E}$ is continuous from above: For $X \in \mathscr{L}^\infty(\Omega, \mathscr{F})$, let

$$\overline{\mathcal{E}}(X) := \sup \left\{ \inf_{n\in\mathbb{N}} \mathcal{E}(X_n) \,\middle|\, (X_n)_{n\in\mathbb{N}} \subset M,\ X_n \geq X_{n+1},\ \inf_{n\in\mathbb{N}} X_n \leq X \right\}.$$

   - Inspired by Choquet's theorem on capacitability and outer measures,
   - Again, linked to superhedging ($\approx$ NFLVR, Delbaen-Schachermayer (1994)),
   - Preserves convexity and sublinearity,
   - Uniqueness (in a certain sense) and representation in terms of countably additive measures.

Stochastic Processes under nonlinear expectations

The extension procedures from the last slide can be used to derive an imprecise version Kolmogorov's theorem on the existence of stochastic processes. This reduces the existence of Markov process to certain properties of a family of transition operators (so-called regular kernels) $(\mathcal{E}_{s,t})_{0\leq s<t}$. The construction of the transition operators is inspired by Nisio (1976). Main examples are:

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- A Brownian Motion with imprecise drift $\mu \in [\underline{\mu}, \overline{\mu}]$ ($\approx$ BSDEs, El Karoui-Peng-Quenez (1997), Coquet et al. (2002))
- A Brownian Motion with imprecise volatility $\sigma \in [\underline{\sigma}, \overline{\sigma}]$ ($\approx$ 2BSDEs, Peng (2007, 2008), Denis-Hu-Peng (2011), Soner-Touzi-Zhang (2011a, 2011b))
- Lévy Processes with imprecise Lévy triplet (Hu-Peng (2009), Neufeld-Nutz (2014), Hollender (2016), Kühn (2018), Denk-Kupper-N. (2017))
- Discrete/Continuous-time Markov chains (Hartfiel (1998), De Cooman-Hermans-Quaeghebeur (2009), Škulj (2009, 2015), Krak-De Bock-Siebes (2017), N. (2018))
- Imprecise Ornstein-Uhlenbeck processes and Geometric Brownian Motions (Epstein-Ji (2013), Vorbrink (2014), Röckner-N. (2019))

Contents

Q-matrices

Definition

A matrix $q = (q_{ij})_{1 \le i,j \le d} \in \mathbb{R}^{d \times d}$ is called a *Q-matrix* if it satisfies the following:

(i) $q_{ii} \le 0$ for all $i \in \{1, \dots, d\}$,

(ii) $q_{ij} \ge 0$ for all $i, j \in \{1, \dots, d\}$ with $i \ne j$,

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We say that a (possibly nonlinear) map $\mathscr{Q}\colon \mathbb{R}^d \to \mathbb{R}^d$ satisfies the *positive maximum principle (PMP)* if for $v = (v_1, \ldots, v_d)^T \in \mathbb{R}^d$ and $i \in \{1, \ldots, d\}$ the following implication holds:

$$v_i = \max_{j=1,\ldots,d} v_j \quad \Longrightarrow \quad (\mathscr{Q}v)_i \leq 0$$

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$\longrightarrow$ $q \in \mathbb{R}^{d\times d}$ is a Q-matrix if and only if it satisfies the PMP and $1 \in \ker q$.

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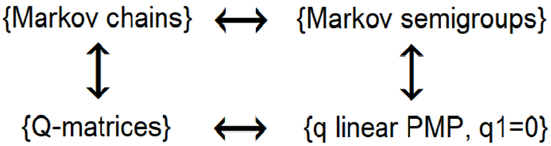
Overview



{Markov chains}
{Markov semigroups}
{Q-matrices}
{q linear PMP, q1=0}

Q-operators: A generalization of Q-matrices

We now want to generalize the concept of a Q-matrix to a nonlinear setup.

Definition (Just for the sublinear case)

A (possibly nonlinear) map $\mathscr{Q}\colon \mathbb{R}^d \to \mathbb{R}^d$ is called a *Q-operator* if the following conditions are satisfied:

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Example: For $d = 3$ and $0 < \underline{\sigma} \leq \overline{\sigma}$ we consider the mapping

$$\mathscr{Q}v := \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \frac{\sigma^2}{2} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} v \approx \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \frac{\sigma^2}{2} \partial_{xx} v$$

Theorem (Main result, the sublinear case)

Let $\mathscr{Q}\colon \mathbb{R}^d \to \mathbb{R}^d$ *be a mapping. Then the following statements are equivalent:*

(i) $\mathscr{Q}$ *is a sublinear Q-operator.*

(ii) $\mathscr{Q}$ *is sublinear, satisfies the PMP and* $\mathscr{Q}(m1) = 0$ *for all* $m \in \mathbb{R}$.

(iii) *There exists a set* $\mathcal{P} \subset \mathbb{R}^{d\times d}$ *of Q-matrices such that*

$$\mathscr{Q}u_0 = \sup_{q\in\mathcal{P}} qu_0 \quad \text{for all } u_0 \in \mathbb{R}^d.$$

(iv) *There exists a sublinear Markov semigroup* $\big(S(t)\big)_{t\geq 0}$ *such that* $u(t) := S(t)u_0$ *defines the unique solution* $u \in C^1\big([0,\infty);\mathbb{R}^d\big)$ *to the initial value problem*

$$u'(t) = \mathscr{Q}u(t) \quad \text{for all } t \geq 0, \qquad u(0) = u_0. \tag{ODE}$$

(v) *There exists a sublinear Markov chain* $\big(\Omega, \mathscr{F}, \mathcal{E}, (X_t)_{t\geq 0}\big)$ *such that*

$$\mathscr{Q}u_0 = \lim_{h\searrow 0} \frac{\mathcal{E}\big(u_0(X_h)\big) - u_0}{h} \quad \text{for all } u_0 \in \mathbb{R}^d.$$

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- Solutions to (ODE) remain bounded. Therefore, a Picard iteration can be used for numerical computations and the convergence rate (depending on the size of the initial value u_0) can be explicitly computed. Other numerical methods such as Runge-Kutta methods can also be applied.

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- Solutions to (ODE) remain bounded. Therefore, a Picard iteration can be used for numerical computations and the convergence rate (depending on the size of the initial value u_0) can be explicitly computed. Other numerical methods such as Runge-Kutta methods can also be applied.
- By solving (ODE) (for example with Euler's method), we can compute "prices" for European contingent claims of the form

$$u(t) = \mathcal{E}(u_0(X_t))$$

under model uncertainty. More precisely,

$$\mathcal{E}(u_0(X_t)) \approx \left(I + \frac{t}{n}\mathcal{Q}\right)^n u_0.$$

Thank you very much for your attention and see you at the poster! :-)