Random Set Solutions to Stochastic Wave Equations

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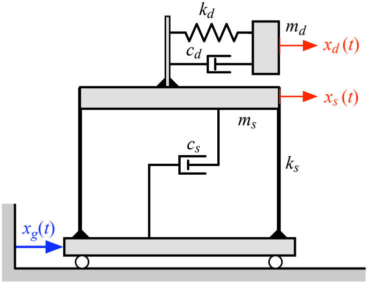
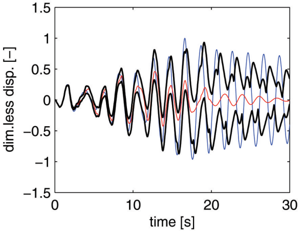
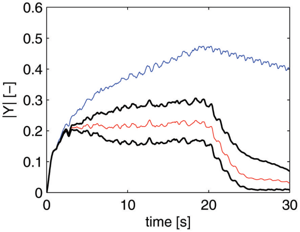


$$\mathbf{M}\begin{bmatrix}\ddot{x}_\mathrm{s}\\ \ddot{x}_\mathrm{d}\end{bmatrix}+\mathbf{C}\begin{bmatrix}\dot{x}_\mathrm{s}\\ \dot{x}_\mathrm{d}\end{bmatrix}+\mathbf{K}\begin{bmatrix}x_\mathrm{s}\\ x_\mathrm{d}\end{bmatrix}=\begin{bmatrix}-1\\ -\frac{m_\mathrm{d}}{m_\mathrm{s}}\end{bmatrix}\ddot{x}_\mathrm{g}$$

Stochastic excitation $\ddot{x}_\mathrm{g}$
Interval-valued coefficients in $\mathbf{C}, \mathbf{K}$
Response is a set-valued process

Interval valued trajectory and interval means w/o TMD:





Known Example – Elastically Bedded Beam

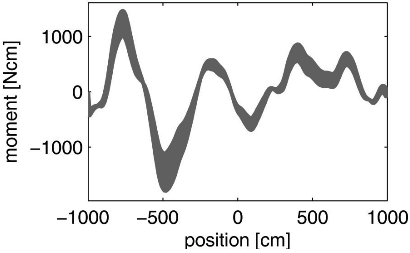
Figure: a buried pipeline.

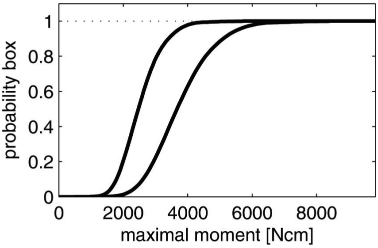
See V. Bolotin, Statistical Methods in Structural Mechanics. San Francisco: Holden-Day 1969, § 61.

$$EI\, w''''(x) + bc\, w(x) = q(x)$$

Load $q(x)$ is a random field
Bedding parameter bc is an interval
Response is a set-valued process

Interval trajectory of bending moment, p-box for maximal bending moment:





New: SPDEs, the Stochastic Wave Equation

The **linear stochastic wave equation** as a prototype of an SPDE:

$$\begin{cases} \partial_t^2 u_c - c^2 \Delta u_c = \dot{W}, \quad x \in \mathbb{R}^d, t \geq 0 \\ u_c | \{t < 0\} = 0 \end{cases}$$

The Laplacian: $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2$.

Space-time white noise excitation $\dot{W}$.

The solution process $u_c = u_c(x, t, \omega)$.

Target: Uncertain propagation speed c as an interval $[\underline{c}, \overline{c}]$.

Applications:

Acoustic waves in a medium under noisy disturbances.

Membrane under noisy excitation.

"A drum in the rain".

Random Set Solutions of SPDEs

Probability space (Ω, Σ, P). **White noise** is a generalized stochastic process with values in the space of distributions

$$\Omega \to \mathcal{D}'(\mathbb{R}^{d+1}), \qquad \omega \to \dot{W}(\omega)$$

The **solution** $\omega \to u_c(x, t, \omega)$ is a **stochastic process** with values in

- $\mathcal{C}(\mathbb{R}^2)$, $d = 1$ (classical)
- $\mathcal{D}'(\mathbb{R}^{d+1})$, $d \geq 2$ (generalized)

Resulting multifunction:

$$U(\omega) = \{u_c(\omega) : c \in [\underline{c}, \overline{c}]\}$$

with values in the power set of $\mathcal{C}(\mathbb{R}^2)$, respectively $\mathcal{D}'(\mathbb{R}^{d+1})$.

Question: Is U a **random set**? Implied by measurability of all

$$U^-(B) = \{\omega \in \Omega : X(\omega) \cap B \neq \emptyset\}$$

where B is any Borel subset of $\mathcal{C}(\mathbb{R}^2)$, respectively $\mathcal{D}'(\mathbb{R}^{d+1})$.

The classical case $d = 1$:

The map $c \to u_c(\omega)$ is continuous with values in $\mathcal{C}(\mathbb{R}^2)$.

The image of $U(\omega)$ of $[\underline{c}, \overline{c}]$ is compact.

Take a dense countable subset $c_1, c_2, \ldots$ of $[\underline{c}, \overline{c}]$.

The sequence $u_{c_n}(\omega)$ is dense in $U(\omega)$ for every ω.

Let O be an open subset of $\mathbb{E}$. Then

$$U^-(O) = \{\omega : U(\omega) \cap O \neq \emptyset\} = \bigcup_{n=1}^{\infty} \{\omega : u_{c_n}(\omega) \in O\}$$

is measurable.

$\mathcal{C}(\mathbb{R}^2)$ is a **Polish space** (metrizable, complete, separable).

By the **Fundamental Measurability Theorem**, U is a **random set** in $\mathcal{C}(\mathbb{R}^2)$.

The generalized case $d \geq 2$:

Same argument, but $\mathcal{D}'(\mathbb{R}^{d+1})$ is **not a Polish space**.

ANNOUNCEMENT 1:

A **new measurability theorem for multifunctions with values in dual spaces** such as $\mathcal{D}'(\mathbb{R}^{d+1})$.

U is a random set also in space dimension $d \geq 2$.

ANNOUNCEMENT 2:

Computation of upper and lower probabilities of intervals (a, b) of the set-valued solution $U(x, t)$ at (x, t) in $d = 1$, e.g.,

$$\overline{P}(a, b)) = P\left(U(x, t) \cap (a, b) \neq \emptyset\right)$$

This employs the observation that

$$(r, \omega) \to v_r(\omega) = \tfrac{2}{t} u_{1/r}(x, t, \omega), \quad r > 0, \qquad v_0(\omega) = 0$$

is a Brownian motion.