Dilation and Asymmetric Relevance

Gregory Wheeler
A. Paul Pedersen

HMI
Human & Machine Intelligence
Frankfurt School of Finance & Management


Frankfurt School

This paper repairs characterization results in (Pedersen and Wheeler 2014; Pedersen and Wheeler 2015).

Dilation

Let $\mathcal{B}$ be a positive measurable partition of Ω.

Say that $\mathcal{B}$ *dilates* A if each $B \in \mathcal{B}$:

$$\underline{P}(A \mid B) \;<\; \underline{P}(A) \;\leqslant\; \overline{P}(A) \;<\; \overline{P}(A \mid B).$$

Dilation

Let $\mathcal{B}$ be a positive measurable partition of Ω.

Say that $\mathcal{B}$ *dilates* A if each $B \in \mathcal{B}$:

$$\underline{P}(A \mid B) \; < \; \underline{P}(A) \; \leqslant \; \overline{P}(A) \; < \; \overline{P}(A \mid B).$$

In other words, $\mathcal{B}$ dilates A just in case the closed interval $\left[\, \underline{P}(A), \quad \overline{P}(A) \,\right]$ is contained within the open interval $\left(\, \underline{P}(A \mid B), \; \overline{P}(A \mid B) \,\right)$ for each $B \in \mathcal{B}$.

Dependence

Given a probability function p on algebra $\mathcal{A}$ and events $A, B \in \mathcal{A}$, define:

$$\mathrm{S}_p(A, B) := \begin{cases} \dfrac{p(A \cap B)}{p(A)p(B)} & \text{if } p(A)p(B) > 0; \\ 1 & \text{otherwise.} \end{cases}$$

Thus the quantity S_p is an index of deviation from stochastic independence between events.

Neighborhoods

Given lower probability space $(\Omega, \mathcal{A}, \mathbb{P}, \underline{P})$, events $A, B \in \mathcal{A}$ with $\underline{P}(B) > 0$, and $\epsilon > 0$, define:

$$\underline{\mathbb{P}}(A \mid B, \epsilon) \;:=\; \{p \in \mathbb{P} : \; |p(A \mid B) - \underline{P}(A \mid B)| \;<\; \epsilon\};$$
$$\overline{\mathbb{P}}(A \mid B, \epsilon) \;:=\; \{p \in \mathbb{P} : \; |p(A \mid B) - \overline{P}(A \mid B)| \;<\; \epsilon\}.$$

Neighborhoods

Given lower probability space $(\Omega, \mathcal{A}, \mathbb{P}, \underline{P})$, events $A, B \in \mathcal{A}$ with $\underline{P}(B) > 0$, and $\epsilon > 0$, define:

$$\underline{\mathbb{P}}(A \mid B, \epsilon) := \{p \in \mathbb{P} : |p(A \mid B) - \underline{P}(A \mid B)| < \epsilon\};$$
$$\overline{\mathbb{P}}(A \mid B, \epsilon) := \{p \in \mathbb{P} : |p(A \mid B) - \overline{P}(A \mid B)| < \epsilon\}.$$

Call the sets $\underline{\mathbb{P}}(A \mid B, \epsilon)$ and $\overline{\mathbb{P}}(A \mid B, \epsilon)$ **lower** and **upper neighborhoods** of A *conditional* on B, respectively, with radius ϵ.

Characterization

Corollary 5.2 of (Pedersen and Wheeler 2014)

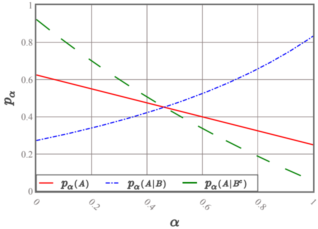
$\mathcal{B}$ dilates A just in case there is $(\epsilon_B)_{B \in \mathcal{B}} \in \mathbb{R}_+^{\mathcal{B}}$ such that

$$\underline{\mathbb{P}}(A \mid B, \epsilon_B) \subseteq \mathsf{S}^-(A, B)$$

and

$$\overline{\mathbb{P}}(A \mid B, \epsilon_B) \subseteq \mathsf{S}^+(A, B).$$

Characterization



Example by Michael Nielsen and Rush Stewart

Corollary 5.2 of (Pedersen and Wheeler 2014)

$\mathcal{B}$ dilates A just in case there is $(\epsilon_B)_{B \in \mathcal{B}} \in \mathbb{R}_+^{\mathcal{B}}$ such that

$$\underline{\mathbb{P}}(A \mid B, \epsilon_B) \subseteq \mathsf{S}^-(A, B)$$

and

$$\overline{\mathbb{P}}(A \mid B, \epsilon_B) \subseteq \mathsf{S}^+(A, B).$$

Relevance

S_p

$$\mathsf{S}_p(A, B) := \frac{p(A \cap B)}{p(A)p(B)}$$

$\underline{\mathsf{S}}_p$ AND $\overline{\mathsf{S}}_p$

$$\underline{\mathsf{S}}_p(A, B) := \frac{p(A \cap B)}{\underline{P}(A)p(B)}$$

and

$$\overline{\mathsf{S}}_p(A, B) := \frac{p(A \cap B)}{\overline{P}(A)p(B)}$$

Relevance

S_p

$$\mathsf{S}_p(A,B) := \frac{p(A \cap B)}{p(A)p(B)}$$

$\underline{\mathsf{S}}_p$ AND $\overline{\mathsf{S}}_p$

$$\underline{\mathsf{S}}_p(A,B) := \frac{p(A \cap B)}{\underline{P}(A)p(B)}$$

and

$$\overline{\mathsf{S}}_p(A,B) := \frac{p(A \cap B)}{\overline{P}(A)p(B)}$$

$$\begin{aligned}
\mathsf{S}^+_{\mathbb{P}}(A,B) &:= \{p \in \mathbb{P} : \mathsf{S}_p(A,B) > 1\};\\
\mathsf{S}^-_{\mathbb{P}}(A,B) &:= \{p \in \mathbb{P} : \mathsf{S}_p(A,B) < 1\};\\
\mathsf{I}_{\mathbb{P}}(A,B) &:= \{p \in \mathbb{P} : \mathsf{S}_p(A,B) = 1\}.
\end{aligned}$$

$$\begin{aligned}
\overline{\mathsf{S}}^+_{\mathbb{P}}(A,B) &:= \{p \in \mathbb{P} : \overline{\mathsf{S}}_p(A,B) > 1\};\\
\underline{\mathsf{S}}^-_{\mathbb{P}}(A,B) &:= \{p \in \mathbb{P} : \underline{\mathsf{S}}_p(A,B) < 1\}.
\end{aligned}$$

Theorem

Let A be an event and $\mathcal{B} = (B_i)_{i \in I}$ be a positive measurable partition for a given set of probability functions $\mathbb{P}$ over an algebra. The following statements are equivalent

(i) $\mathcal{B}$ *dilates* A*;*

(ii) There exists $\epsilon > 0$ *such that for every* $i \in I$*:*

$$\underline{\mathbb{P}}(A \mid B_i, \epsilon) \subseteq \underline{\mathsf{S}}^{-}_{\mathbb{P}}(A, B_i) \text{ and } \overline{\mathbb{P}}(H \mid B_i, \epsilon) \subseteq \overline{\mathsf{S}}^{+}_{\mathbb{P}}(A, B_i)$$

References

Pedersen, A. P. and G. Wheeler (2014).
Demystifying dilation.
Erkenntnis 79(6), 1305–1342.

Pedersen, A. P. and G. Wheeler (2015).
Dilation, disintegrations, and delayed decisions.
In *Proceedings of the 9th Symposium on Imprecise Probabilities and Their Applications (ISIPTA)*, Pescara, Italy, pp. 227–236.