Extensions of Sets of Markov Operators Under Epistemic Irrelevance

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Markov operators

Definition

Let $\mathcal{H}$ and $\mathcal{K}$ be sets of gambles. A Markov operator is a linear operator $T\colon \mathcal{H} \to \mathcal{K}$, such that:

1. if $f \le g$ then $Tf \le Tg$;
2. $T1_{\mathcal{X}} = 1_{\mathcal{X}}$.

A set of Markov operators $\mathcal{T}$ can specify a probabilistic model by requiring:

1. if f is a desirable gamble, then Tf is also desirable **for every** $T \in \mathcal{T}$;
2. if f is undesirable, then $T \in \mathcal{T}$ **exists** such that Tf is undesirable.

A set of (almost) desirable gambles $\mathcal{D}$ satisfying the above requirements is said to be generated by $\mathcal{T}$.

In general, multiple such sets exist.

Examples

Sets of Markov operators are can be combined to achieve certain properties of probabilistic models.

Some motivating examples:

1. Conditional expectation $E_p(f|\mathcal{B})$ of a desirable gambles f is usually also considered desirable. (If $p \in \mathcal{M}$ and $\mathcal{B} = \{\emptyset, \mathcal{X}\}$, this model is equivalent to the model given by the credal set $\mathcal{M}$.)
2. Symmetry with respect to elements of $\mathcal{Y}$ for the gambles

$$f\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$$

can be modelled by requiring that the set of desirable gambles is generated by the set $\mathcal{P}(\mathcal{Y})$ of all operators T_σ of the form

$$[P_\sigma f](x, y) = f(x, \sigma(y)),$$

where σ is a permutation on $\mathcal{Y}$.

Extension of imprecise probabilistic models

We consider specific question of extending a probabilistic model on $\mathcal{X}$ to a lager space, such as $\mathcal{X} \times \mathcal{Y}$ (or $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$).

	set	gambles	desirable gambles	operators
space	$\mathcal{X}$	$\mathcal{G}(\mathcal{X})$	$\mathcal{D}(\mathcal{X})$	$\mathcal{T}$
extended space	$\mathcal{X} \times \mathcal{Y}$	$\mathcal{G}(\mathcal{X} \times \mathcal{Y})$	$\mathcal{D}(\mathcal{X} \times \mathcal{Y})$	$\tilde{\mathcal{T}} = ?$

How can we construct an extension $\tilde{\mathcal{T}}$ of $\mathcal{T}$ that generates $\mathcal{D}(\mathcal{X} \times \mathcal{Y})$?

The extension $\tilde{\mathcal{T}}$ depends on the type of extension $\mathcal{D}(\mathcal{X} \times \mathcal{Y})$ of $\mathcal{D}(\mathcal{X})$.

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Example – strong products

Take for example the strong product of credal sets $\mathcal{M}$ and $\mathcal{N}$ on separate spaces $\mathcal{X}$ and $\mathcal{Y}$:

	set	credal sets	operators
space	$\mathcal{X}, \mathcal{Y}$	$\mathcal{M}, \mathcal{N}$	$\mathcal{T}, \mathcal{S}$
extended space	$\mathcal{X} \times \mathcal{Y}$	$\mathcal{M} \times \mathcal{N}$	$\mathcal{T} \otimes \mathcal{S}$

where

$$\mathcal{T} \otimes \mathcal{S} = \{T \otimes S \colon T \in \mathcal{T}, S \in \mathcal{S}\}$$

and $\otimes$ denotes the tensor product of operators.

Note:

The structure of the model, i.e. independent product, must be pre-specified – not every $\mathcal{T} \otimes \mathcal{S}$-generated credal set is an independent product of its marginals.

Epistemic irrelevance

- A set of desirable gambles $\mathcal{D} \subset \mathcal{G}(\mathcal{X} \times \mathcal{Y})$ satisfies epistemic irrelevance $\mathcal{Y} \to \mathcal{X}$ if for every $\mathcal{X}$-measurable gamble f the following conditions are equivalent:
  - $f \in \mathcal{D}$;
  - $I_y f \in \mathcal{D}$ for every $y \in \mathcal{Y}$.

Remark

A gamble f is $\mathcal{X}$-measurable if $f(x, y) = f(x, y')$ for every $y, y' \in \mathcal{Y}$.

- Epistemic irrelevance combines two properties:
  - symmetry w.r.t. $\mathcal{Y}$;
  - the sum $f = \sum_{y \in \mathcal{Y}} I_y f$ is only desirable if all $I_y f$ are desirable.

Example – symmetry alone does not imply irrelevance

Let $\mathcal{X} = \mathcal{Y} = \{0, 1\}$.
Consider the following gambles:

f:

	$Y = 0$	$Y = 1$
$X = 0$	-1	-1
$X = 1$	1	1

, which is $\mathcal{X}$-measurable;

$I_{(Y=0)}f$:

	$Y = 0$	$Y = 1$
$X = 0$	-1	0
$X = 1$	1	0

, $I_{(Y=1)}f$:

	$Y = 0$	$Y = 1$
$X = 0$	0	-1
$X = 1$	0	1

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Take two probability mass functions:

$$p_1: \begin{array}{c|cc} & Y=0 & Y=1 \\ \hline X=0 & 3/16 & 3/16 \\ X=1 & 2/16 & 8/16 \end{array}, \quad p_2: \begin{array}{c|cc} & Y=0 & Y=1 \\ \hline X=0 & 3/16 & 3/16 \\ X=1 & 8/16 & 2/16 \end{array},$$

Let $\mathcal{M} = \{p_1, p_2\}$ and then

$$\underline{P}_{\mathcal{M}}(f) = 4/16, \underline{P}_{\mathcal{M}}(I_{(Y=0)}f) = -1/16, \underline{P}_{\mathcal{M}}(I_{(Y=1)}f) = -1/16.$$

So we have that

- $I_{(Y=0)}f$ and $I_{(Y=1)}f$ are undesirable,
- while $f = I_{(Y=0)}f + I_{(Y=1)}f$ is desirable.

Thus, even if there is clear symmetry with respect to $\mathcal{Y}$, we do not have epistemic irrelevance.

Additive independent extension

In addition to symmetry we need a new property to guarantee epistemic irrelevance:

Definition (Additive independent extension)

A set $\mathcal{D}$ is an additive independent extension of $\{\mathcal{D}_i \subset \mathcal{G}_i\}_{i\in I}$ if

- $f \in \mathcal{D}$;
- $f = \sum_{i\in I} f_i$, $f_i \in \mathcal{G}_i$

imply that $\exists i \in I\colon f_i \in \mathcal{D}_i$.

A sum of undesirable gambles cannot be desirable.

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Epistemic irrelevant additive independent extension

- Let $\mathcal{T}$ be a set of Markov operators on $\mathcal{G}(\mathcal{X})$.
- We construct the following set of Markov operators:

$$\tilde{\mathcal{T}} = \left\{ \tilde{T} \colon \tilde{T} f = \sum_{y \in \mathcal{Y}} I_y T_y f(\cdot, y), T_y \in \mathcal{T} \, \forall y \in \mathcal{Y} \right\}.$$

- Restricted to $G(\mathcal{X}|y)$ that is isomorphic to $\mathcal{G}(\mathcal{X}), \tilde{\mathcal{T}}$ acts as $\mathcal{T}$.
- Hence, the generated sets of desirable gambles $\mathcal{D}(\mathcal{X}|y)$ are all generated by $\mathcal{T}$, yet are not necessarily equal, as $\mathcal{T}$ can have multiple generated models.
- So, we additionally need to require symmetry, which we can do by adding the set of permutation operators.

Main result

Let $\mathcal{D}$ be a set of desirable gambles that is an additive independent extension of $\{\mathcal{D} \cap \mathcal{G}(\mathcal{X}|y) : y \in \mathcal{Y}\}$.
Then the following are equivalent:

1. $\mathcal{D}$ satisfies epistemic irrelevance $\mathcal{Y} \to \mathcal{X}$;
2. $\mathcal{D}$ is generated by $\tilde{\mathcal{T}} \cup \mathcal{P}(\mathcal{Y})$;
3. two sets of desirable gambles
   - $\mathcal{D}(\mathcal{X})$: generated by $\mathcal{T}$ and
   - $\mathcal{D}(\mathcal{Y})$: generated by $\mathcal{P}(\mathcal{Y})$ (only required to be symmetric w.r.t. $\mathcal{Y}$)

   exist so that every $f \in \mathcal{D}$ can be written in the form:

$$f = f_{\mathcal{Y}} + \sum_{y \in \mathcal{Y}} I_y f_y,$$

where $f_{\mathcal{Y}} \in \mathcal{D}(\mathcal{Y})$ and $f_y \in \mathcal{D}(\mathcal{X})$ for every $y \in \mathcal{Y}$.

Further work

- The advantage of operators approach is that the probabilistic model (even imprecise) does not need to be fully specified – we can only have a set of (local) conditional models, which we can easily extend to larger (global) probability spaces.
- Moreover, additional requirements, such as symmetry, can easily be given in terms of additional sets of operators.
- This can be useful in models where compatible probabilistic models need to be constructed based on conditional models only.
- Such examples include stochastic processes, credal networks, probability trees and others.
- In particular, it would be interesting to impose requirements such as time homogeneity or Markov property in terms of Markov operators.