A Cantelli-type inequality for constructing non-parametric p-boxes based on exchangeability

Matthias C. M. Troffaes Tathagata Basu

Department of Mathematical Sciences
Durham University, UK

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Outline

Cantelli's inequality & induced p-box

- random variable X, known mean μ and variance σ^2
- Cantelli's inequality [1]:

$$0 \leq P\left(\frac{X-\mu}{\sigma} \leq \lambda\right) \leq \frac{1}{1+\lambda^2} \qquad \text{if } \lambda \leq 0, \qquad (1a)$$

$$\frac{\lambda^2}{1+\lambda^2} \leq P\left(\frac{X-\mu}{\sigma} \leq \lambda\right) \leq 1 \qquad \text{if } \lambda \leq 0. \qquad (1b)$$

- induces a p-box (lower & upper cdf, bounding a set of probability measures)

Issue

What if only sample mean and sample standard deviation are known?

Contributions: Problem Statement

Assumptions

$X_1, \ldots, X_n, X_{n+1}$ is a finite sequence of discrete exchangeable random variables.

Notation

- $\overline{X} := \frac{1}{n} \sum_{j=1}^{n} X_j$
- $S^2 := \frac{\sum_{j=1}^{n} (X_j - \overline{X})^2}{(n-1)}$

Objective

Find functions $\underline{f}$ and $\overline{f}$ such that

$$\underline{f}(\lambda, n) \leq P\left(\frac{X_{n+1} - \overline{X}}{S} \leq \lambda\right) \leq \overline{f}(\lambda, n) \tag{2}$$

Contributions: Cantelli-type inequality

Theorem

For every $\lambda \geq 0$,

$$\frac{1}{n+1}\left\lceil \frac{(n+1)\lambda_n^2}{\lambda_n^2+1} \right\rceil \leq P\left(\frac{X_{n+1}-\overline{X}}{S+\frac{\Delta_n}{\sqrt{n}}} < \lambda \right) \leq 1 \tag{3}$$

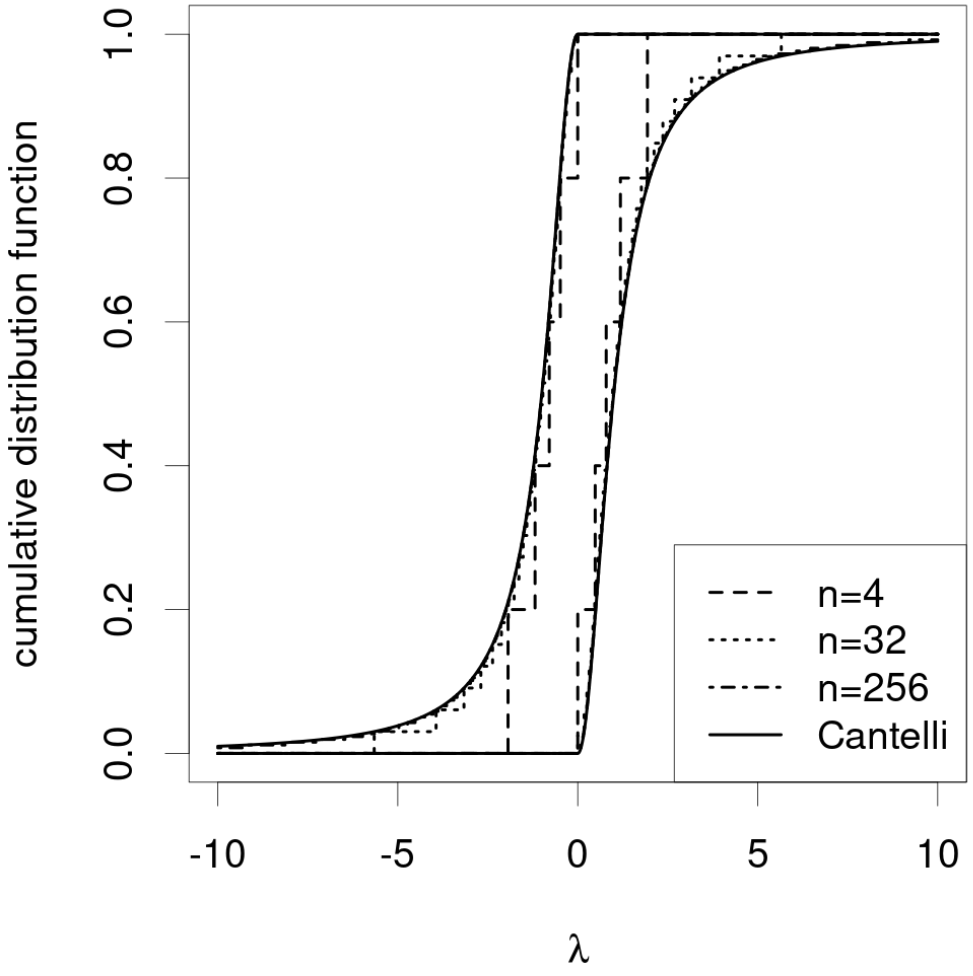
where $\lambda_n := \frac{n}{\sqrt{n^2-1}}\lambda$ *and* $\Delta_n := \sqrt{\frac{n+1}{n-1}}(\max X_j - \min X_j)$.
Similarly, for every $\lambda \leq 0$,

$$0 \leq P\left(\frac{X_{n+1}-\overline{X}}{S+\frac{\Delta_n}{\sqrt{n}}} \leq \lambda \right) \leq \frac{1}{n+1}\left\lfloor \frac{n+1}{\lambda_n^2+1} \right\rfloor. \tag{4}$$

Here, $\lfloor x \rfloor := \max\{n \in \mathbb{Z} \colon n \leq x\}$ and $\lceil x \rceil := -\lfloor -x \rfloor$.

Contributions: Cantelli-type inequality

Plotting left and right hand sides from inequalities in theorem:



Contributions: P-box and Prediction Interval

Inequalities from theorem induce the following:

Non-parametric p-box

. . . on the random variable $Z_{n+1} = \frac{X_{n+1} - \overline{X}}{S + \frac{\Delta_n}{\sqrt{n}}}$.

Not a p-box on X_{n+1} directly!

Prediction interval

. . . on the random variable X_{n+1}.
For any ℓ_1 and ℓ_2, we can calculate α_1 and α_2 such that

$$\alpha_1 \leq P(\overline{X} - \ell_1 S_n < X_{n+1} \leq \overline{X} + \ell_2 S_n) \leq \alpha_2 \tag{5}$$

where, $S_n := S + \frac{\Delta_n}{\sqrt{n}}$.

Conclusions I

- novel Cantelli-type inequality
- induces non-parametric p-box and prediction interval
- only assumes exchangeability (rather than conditional independence)
- only uses sample mean and sample standard deviation
- similar to Saw [2] (but Saw does not induce a p-box)
- useful for modelling when only sample mean and sample standard deviation are known (e.g. measurement problems)

Conclusions II

- p-box only on Z_{n+1} and not on X_{n+1}
- use of prediction interval not entirely clear
- p-box on X_{n+1} , conditional on $\overline{X}$ and S, using exchangeability, remains and open problem
  (might be impossible as pointed out by a kind reviewer)

Thank you for your attention!

A Cantelli-type inequality for constructing non-parametric p-boxes based on exchangeability

Matthias C. M. Troffaes and Tathagata Basu
Department of Mathematical Sciences, Durham University, UK

Introduction

- We derive a Cantelli-type inequality to produce a non-parametric p-box and prediction interval.
- Based on sample mean and sample standard deviation only (i.e. no parametric assumptions).
- We assume exchangeability (rather than conditional independence).
- Useful for modelling when only sample mean and sample standard deviation are known (e.g. measurement problems).

Exchangeability [3]

We say a finite sequence $X_1, X_2, \ldots, X_n$ of discrete random variables is exchangeable if, for all $\sigma \in \Sigma_n$ and all $x_1, x_2, \ldots, x_n \in \mathbb{R}$

$$P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n) = P(X_1 = x_{\sigma(1)}, \cdots, X_n = x_{\sigma(n)})$$

Independent $\implies$ Exchangeable

Cantelli's inequality [1] *and p-boxes* [2]

Let X be a random variable with known mean μ and known variance σ^2. Then the Cantelli's inequality is given by:

$$0 \leq P\left(\frac{X-\mu}{\sigma} \leq \lambda\right) \leq \frac{1}{1+\lambda^2} \qquad \text{if } \lambda \leq 0,$$
$$\frac{\lambda^2}{1+\lambda^2} \leq P\left(\frac{X-\mu}{\sigma} \leq \lambda\right) \leq 1 \qquad \text{if } \lambda \leq 0.$$

A p-box is specified by two cumulative distribution functions $\underline{F}$ and $\overline{F}$, and represents the set of all cumulative distribution functions bounded by $\underline{F}$ and $\overline{F}$:

$$\{F \in \mathcal{F} \colon \underline{F}(x) \leq F(x) \leq \overline{F}(x), \forall x \in \mathbb{R}\}.$$

Cantelli's inequality gives distribution free p-box for known μ and σ^2:

Cantelli's inequality can also be written as a p-box on X:

$$0 \leq P(X \leq x) \leq \frac{\sigma^2}{\sigma^2 + (x-\mu)^2} \qquad \text{if } x \leq \mu$$
$$\frac{(x-\mu)^2}{\sigma^2 + (x-\mu)^2} \leq P(X \leq x) \leq 1 \qquad \text{if } x \geq \mu.$$

Saw's inequality (Chebyshev) [4]

Let $X_1, \ldots, X_n$, X_{n+1} be a sequence of discrete exchangeable random variables. Define $\overline{X} := \frac{1}{n}\sum_{j=1}^{n} X_j$, $S^2 := \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \overline{X})^2$ and $Q^2 := \frac{n+1}{n}S^2$. Then for every $\lambda \geq 1$

$$P\left(|X_{n+1} - \overline{X}| > \lambda Q\right) \leq \frac{1}{n+1}\left\lfloor \frac{n+1}{\lambda_n^2} \right\rfloor_*$$

where $\lambda_n := \sqrt{\frac{n\lambda^2}{n-1+\lambda^2}}$ and $\lfloor x \rfloor_* := \max\{n \in \mathbb{Z} \colon n < x\}$.

Main Result

Let, $X_1, \ldots, X_n$, X_{n+1} be a finite sequence of discrete exchangeable random variables. Let $\Delta \in \mathbb{R}$ denote the range of the X_j i.e. $\Delta := \max X_j - \min X_j$ where $\max X_j$ is the maximum value that can be attained by X_j, and $\min X_j$ is the minimum value. Let $\overline{X} := \sum_{j=1}^{n} X_j/n$, and $S^2 := \sum_{j=1}^{n}(X_j - \overline{X})^2/(n-1)$. Then for every $\lambda \geq 0$,

$$\frac{1}{n+1}\left\lceil \frac{(n+1)\lambda_n^2}{\lambda_n^2+1} \right\rceil \leq P\left(\frac{X_{n+1}-\overline{X}}{S+\frac{\Delta_n}{\sqrt{n}}} < \lambda\right) \leq 1 \tag{3}$$

where $\lambda_n := \frac{n}{\sqrt{n^2-1}}\lambda$ and $\Delta_n := \sqrt{\frac{n+1}{n-1}}\Delta$. Similarly, for $\lambda \leq 0$,

$$0 \leq P\left(\frac{X_{n+1}-\overline{X}}{S+\frac{\Delta_n}{\sqrt{n}}} \leq \lambda\right) \leq \frac{1}{n+1}\left\lfloor \frac{n+1}{\lambda_n^2+1} \right\rfloor. \tag{4}$$

Here, $\lfloor x \rfloor := \max\{n \in \mathbb{Z} \colon n \leq x\}$ and $\lceil x \rceil := -\lfloor -x \rfloor$.

P-box and Imprecise Prediction Interval

We use our main result to construct a **p-box** for the random variable:

$$Z_{n+1} = \frac{X_{n+1} - \overline{X}}{S + \frac{\Delta_n}{\sqrt{n}}}$$

However, this cannot be used as a p-box for X_{n+1}. We cannot substitute the observed values for $\overline{X}$ and S in the equation.
Our result allows to construct an asymmetric **prediction interval**. Our result gives, for every ℓ_1 and ℓ_2, the following bounds:

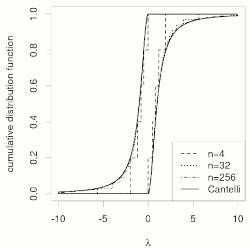
$$\underline{p}_1 \leq P(Z_{n+1} \leq \ell_1) \leq \overline{p}_1$$
$$\underline{p}_2 \leq P(Z_{n+1} \leq \ell_2) \leq \overline{p}_2$$

This gives us the following bounds on the coverage probability:

$$\underline{p}_2 - \overline{p}_1 \leq P(\overline{X} + \ell_1 S_n \leq X_{n+1} \leq \overline{X} + \ell_2 S_n) \leq \overline{p}_2 - \underline{p}_1$$

where, $S_n := S + \frac{\Delta_n}{\sqrt{n}}$.

Illustration



We compare our bounds with Cantelli's bounds. The correction term is larger for smaller sample sizes, which results in tighter bounds for smaller samples. For instance, for $n = 4$ this value is approximately $0.645 \times \Delta$, whereas for $n = 256$ it is only $0.063 \times \Delta$.

Discussion

- We provide a Cantelli type inequality based on exchangeability and construct a non-parametric p-box.
- We propose a prediction interval based on exchangeability.
- We can construct p-box only for the random variable Z_{n+1}.
- Finding bounds on the cumulative distribution of X_{n+1}, conditional on $\overline{X}$ and S using exchangeability remains and open problem.

References

[1] F. P. Cantelli. 'Sui confini della probabilità'. In: Atti del Congresso Internazionale del Matematici 6 (1928). Bologna, pp. 47–59.
[2] Scott Ferson et al. Constructing Probability Boxes and Dempster-Shafer Structures. Tech. rep. SAND2002–4015. Unabridged version. Sandia National Laboratories, Jan. 2003.
[3] J. F. C. Kingman. 'Uses of Exchangeability'. In: The Annals of Probability 6.2 (Apr. 1978), pp. 183–197. DOI: 10.1214/aop/1176995566.
[4] John G. Saw, Mark C. K. Yang and Tse Chin Mo. 'Chebyshev Inequality with Estimated Mean and Variance'. In: The American Statistician 38.2 (1984), pp. 130–132. URL: http://www.jstor.org/stable/2683249.

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We look forward to seeing you at our poster!

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John G. Saw, Mark C. K. Yang, and Tse Chin Mo. "Chebyshev Inequality with Estimated Mean and Variance". In: *The American Statistician* 38.2 (1984), pp. 130–132.