

Two-State Imprecise Markov Chains for Statistical Modelling of Two-State Non-Markovian Processes

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Aim

present a new framework for

- ▶ fitting an imprecise two-state Markov chain
- ▶ to data generated by a two-state non-Markovian process
- ▶ via an imprecise version of MCMC

The Old Way

1. transition times $T_i \sim \text{Exp}(\lambda_i)$
(i.e. precise Markov chain assumed)
2. set of priors for λ_i
3. combine with data to get set of posteriors for λ_i
4. posterior predictive bounds on λ_i
5. use these bounds to fix an imprecise Markov chain

what is wrong with this approach?

The Old Way

what is wrong with this approach?

- ▶ model = set of distributions on parameters of a precise Markov chain
imprecise Markov chains are not equivalent to this model
- ▶ sampling uncertainty ignored: only posterior predictive bounds are used
no full uncertainty quantification
- ▶ imprecision does not reflect violations of stationarity & Markovianity
wasted opportunity
- ▶ non-Bayesian analysis: **inferences are not coherent with the model**

The Newly Proposed Way

1. transition times $T_j \sim f_j(\lambda_j)$
where f_j is a **non-Markovian and/or non-stationary process**
and λ_j are unknown parameters of this process
2. set of prior distributions on λ_j
3. for each prior, use standard MCMC to sample posterior realizations of λ_j
i.e. fit the non-Markovian process to the data
4. for each posterior sample of λ_j **bound process by an imprecise Markov chain**
5. produce sample of posterior bounds for any predictive quantity

The Newly Proposed Way

Nice features of this new approach:

- ▶ a form of **imprecise MCMC**: imprecision built into the predictive part
this works because of Walley's marginal extension theorem
- ▶ **fully coherent** robust Bayesian approach:
predictions are directly derived from posterior distribution
- ▶ imprecision reflects lack of data & non-Markovianity & non-stationarity
(even if there is lot's of data, inferences can remain imprecise)

How Did We Do It

- ▶ non-Markovian behaviour modelled by a multi-state Markov chain can model **any phase-type distribution** for transition times!
(every distribution can be approximated by a phase-type distribution)
- ▶ bounding by **lumping** multi-state Markov chains into imprecise two-state Markov chains
- ▶ details: see poster

Conclusions

- ▶ we've improved the way imprecise Markov chains are fitted
- ▶ imprecision results not only from limited data but also from characteristics of the process
- ▶ principles may apply to fitting of general stochastic processes (not only imprecise Markov chains)
- ▶ we identified a class of problems where imprecise MCMC is easy
- ▶ larger problems may require imprecise ABC
if likelihood of phase-type distribution has no closed form

Thank You For Your Attention

We look forward to see you at our poster!

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1. Aims

We aim to present a new framework for:

- fitting an imprecise two-state Markov chain to data generated by a two-state non-Markovian process
- an imprecise version of MCMC

2. Old approach

- imputation times $T_i \sim \text{Exp}(\lambda_i)$ (i.e. precise Markov chain assumed)
- set of states for A_i
- combine with data to get set of parameters for A_i
- generate predictive bounds as A_i
- use these bounds to fit an imprecise Markov chain

3. What is wrong with the old approach?

- Model: set of distributions on parameters of a precise Markov chain, imprecise Markov chain not considered in this model
- Sampling: completely ignored, only predictive bounds are used, still full accuracy questionable
- Imprecise: does not reflect solutions of optimality & Markovian
- Two-Parameter model: assumes set of unknown only the model

4. New approach

- imputation times $T_i \sim \text{Exp}(\lambda_i)$ where λ_i is a non-Markovian and/or nonstationary process and λ_i unknown parameters of the process
- set of prior distributions on λ_i
- in each prior, use standard MCMC to sample predictive values for A_i
- fit the non-Markovian process to the data
- for each parameter sample of λ_i , generate priors for imprecise Markov chain
- produce a sample of predictive bounds for any predictor quantity

5. Advantages to the new approach

- a more of imprecise MCMC: imprecise model and imprecise data part
- the methods because of Markov's regional extension theorem
- full Markov chain formalism approach
- probabilities are directly derived from posterior distribution
- imprecise: reflects both lack of data & non-Markovianity & use optimality (even if there is lack of data, influences on certain aspects)

6. How did we do it?

Step 1
Model a two-state Markov process by a three-state Markov chain. For example (nonergodic) transition from state 2 to state 1:

Figure 1: Example of a Markov chain

with the following transition rate matrix:

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 \\ \lambda_2 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix}$$

Step 2
Fit the three-state Markov chain to the data through its associated phase-type distribution. Here, the three state data to estimate the parameters $\lambda_1, \lambda_2, \lambda_3$ are $\{y_i\}$:

M=20

M=500

Figure 2: Histogram of times to transition between states (data) for state 1. Left: 20 samples, right: 500 samples, with the right figure being of 200 observations.

- All parameters fitted using MCMC
- The model is implemented in R
- Use automatically chosen direct programming of the phase-type distribution, as well as calculating the fitting time and upper probabilities for being in each the working state (see step 5)

7. Results

Figure 3: Posterior densities for the imprecise model with two parameters (left) and three parameters (right). The vertical lines indicate posterior quantiles.

(1) For small sample size ($M = 20$) (2) For large sample size ($M = 500$)

8. Comparison With Two-State Exponential Model

Figure 4: Posterior densities for the exponential model with two parameters (left) and three parameters (right). The vertical lines indicate posterior quantiles.

(1) For small sample size ($M = 20$) (2) For large sample size ($M = 500$)

D. Conclusions

- We have proposed the new imprecise Markov chain model
- Imprecision can result not only from limited data but also from the uncertainty of the process
- These principles may apply to fitting of general stochastic processes (not only imprecise Markov chains)
- We identified a class of predictors whose imprecise MCMC is easy
- Large real-time map require imprecise MCMC if likelihood of phase-type distribution has to be used

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